ANGULAR MOMENTUM RAISING AND LOWERING OPERATORS FROM RECTANGULAR COORDINATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.8.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

To calculate the eigenfunctions of angular momentum, we will need expressions for the raising and lowering operators $L_{\pm}$ in spherical coordinates. We've seen one way of getting these by working with the gradient in spherical coordinates from the start, but it is also possible to convert the rectangular forms of $L_{\pm}$ to spherical coordinates by using the chain rule from calculus. This method is similar to one we used earlier in 2-d. To set the scene, we need the conversion formulas between rectangular and spherical coordinates:

$$\begin{aligned}
x &= r\sin\theta\cos\phi && (1)\\
y &= r\sin\theta\sin\phi && (2)\\
z &= r\cos\theta && (3)\\
r &= \sqrt{x^2+y^2+z^2} && (4)\\
\theta &= \arctan\frac{\sqrt{x^2+y^2}}{z} && (5)\\
&= \arctan\frac{q}{z} && (6)\\
\phi &= \arctan\frac{y}{x} && (7)
\end{aligned}$$

To simplify the notation, we've defined

$$q\equiv\sqrt{x^2+y^2}=r\sin\theta \tag{8}$$

We'll also use shorthand notation for sines and cosines so that

$$\begin{aligned}
s_\theta &\equiv \sin\theta && (9)\\
c_\theta &\equiv \cos\theta && (10)
\end{aligned}$$

and similarly for ϕ. We'll also use the notation ∂_r to mean the partial derivative with respect to r, with a similar notation for other derivatives.

The required derivatives are

$$\partial_x = \partial_x r \cdot \partial_r + \partial_x \theta \cdot \partial_\theta + \partial_x \phi \cdot \partial_\phi \tag{11}$$
$$\partial_y = \partial_y r \cdot \partial_r + \partial_y \theta \cdot \partial_\theta + \partial_y \phi \cdot \partial_\phi \tag{12}$$
$$\partial_z = \partial_z r \cdot \partial_r + \partial_z \theta \cdot \partial_\theta \tag{13}$$

The required derivatives are

$$\partial_x r = \frac{x}{r} \tag{14}$$
$$\partial_y r = \frac{y}{r} \tag{15}$$
$$\partial_z r = \frac{z}{r} \tag{16}$$
$$\partial_x \theta = \frac{x/q}{z\left(1+\frac{q^2}{z^2}\right)} \tag{17}$$
$$= \frac{xz}{qr^2} \tag{18}$$
$$\partial_y \theta = \frac{yz}{qr^2} \tag{19}$$
$$\partial_z \theta = -\frac{q}{r^2} \tag{20}$$
$$\partial_x \phi = \frac{-y/x^2}{1+y^2/x^2} \tag{21}$$
$$= -\frac{y}{q^2} \tag{22}$$
$$\partial_y \phi = \frac{x}{q^2} \tag{23}$$
$$\partial_z \phi = 0 \tag{24}$$

Plugging all these into 11 to 13 we have

$$\partial_x = \frac{x}{r}\partial_r + \frac{xz}{qr^2}\partial_\theta - \frac{y}{q^2}\partial_\phi \tag{25}$$
$$\partial_y = \frac{y}{r}\partial_r + \frac{yz}{qr^2}\partial_\theta + \frac{x}{q^2}\partial_\phi \tag{26}$$
$$\partial_z = \frac{z}{r}\partial_r - \frac{q}{r^2}\partial_\theta \tag{27}$$

We can now calculate the components L_x and L_y:

$$
\begin{aligned}
L_x &= -i\hbar\left(y\partial_z - z\partial_y\right) && (28)\\
&= -i\hbar\left[\frac{yz}{r}\partial_r - \frac{yq}{r^2}\partial_\theta - \frac{yz}{r}\partial_r - \frac{yz^2}{qr^2}\partial_\theta - \frac{xz}{q^2}\partial_\phi\right] && (29)\\
&= i\hbar\left[\left(\frac{yq}{r^2} + \frac{yz^2}{qr^2}\right)\partial_\theta + \frac{xz}{q^2}\partial_\phi\right] && (30)\\
&= i\hbar\left[\left(s_\theta^2 s_\phi + \frac{s_\theta s_\phi c_\theta^2}{s_\theta}\right)\partial_\theta + \frac{s_\theta c_\theta c_\phi}{s_\theta^2}\partial_\phi\right] && (31)\\
&= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) && (32)\\
L_y &= -i\hbar\left(z\partial_x - x\partial_z\right) && (33)\\
&= -i\hbar\left[\frac{xz}{r}\partial_r + \frac{xz^2}{qr^2}\partial_\theta - \frac{xz}{r}\partial_r + \frac{xq}{r^2}\partial_\theta - \frac{yz}{q^2}\partial_\phi\right] && (34)\\
&= i\hbar\left[\left(-\frac{xz^2}{qr^2} - \frac{xq}{r^2}\right)\partial_\theta + \frac{yz}{q^2}\partial_\phi\right] && (35)\\
&= i\hbar\left[\left(-\frac{s_\theta c_\phi c_\theta^2}{s_\theta} - s_\theta^2 c_\phi\right)\partial_\theta + \frac{s_\theta c_\theta s_\phi}{s_\theta^2}\partial_\phi\right] && (36)\\
&= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) && (37)
\end{aligned}
$$

From this we get the raising and lowering operators

$$
\begin{aligned}
L_\pm &= L_x \pm iL_y && (38)\\
&= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \mp && (39)\\
&\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) && (40)\\
&= \hbar e^{\pm i\phi}\frac{\partial}{\partial\theta} \pm i\hbar e^{\pm i\phi}\cot\theta\frac{\partial}{\partial\phi} && (41)\\
&= \hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) && (42)
\end{aligned}
$$

[Admittedly, it's probably easier and more elegant to use spherical coordinates from the start, but it's instructive to see how it's done starting with rectangular coordinates.]