TOTAL ANGULAR MOMENTUM IS HERMITIAN

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.9.

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The total angular momentum operator L^2 can be written in spherical coordinates as

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad (1)$$

As L^2 is an observable, it should be Hermitian. We can verify this by showing that

$$\left\langle\psi_2\left|L^2\right|\psi_1\right\rangle=\left\langle\psi_1\left|L^2\right|\psi_2\right\rangle^* \quad (2)$$

In spherical coordinates, this becomes

$$\int\psi_2^*\left(L^2\psi_1\right)d\Omega=\left[\int\psi_1^*\left(L^2\psi_2\right)d\Omega\right]^* \quad (3)$$

The element of solid angle $d\Omega=\sin\theta\,d\theta\,d\phi$, so the full integral is

$$\int\psi_2^*\left(L^2\psi_1\right)d\Omega=\int_0^{2\pi}\int_0^{\pi}\psi_2^*\left(L^2\psi_1\right)\sin\theta\,d\theta\,d\phi \quad (4)$$

We can verify 3 by showing that it is true for each of the two terms in 1 separately. As usual for these sorts of integrals, we need to use integration by parts. To simplify things, we'll consider $-L^2/\hbar^2$ so we can deal only with the terms in the brackets in 1. We'll also use the shorthand notation

$$s \equiv \sin\theta \quad (5)$$

$$c \equiv \cos\theta \quad (6)$$

Also, a prime indicates a derivative with respect to θ: $\psi_1'\equiv\frac{\partial\psi_1}{\partial\theta}$, etc.

For the first term, we have, considering only the integration over θ:

$$\int_0^\pi \psi_2^* \frac{1}{s} \frac{\partial}{\partial\theta}\left(s \frac{\partial\psi_1}{\partial\theta}\right) s\, d\theta = \int_0^\pi \left[\psi_2^* c\psi_1' + \psi_2^* s\psi_1''\right] d\theta \tag{7}$$

$$= \psi_2^* c\psi_1\big|_0^\pi + \psi_2^* s\psi_1'\big|_0^\pi - \tag{8}$$

$$\int_0^\pi \left[(\psi_2^*)' c\psi_1 - \psi_2^* s\psi_1\right] d\theta - \tag{9}$$

$$\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta \tag{10}$$

The second term in 8 is zero since $\sin 0 = \sin\pi = 0$, but we can't ignore the first term, which is not, in general, zero. Thus we are left with

$$\int_0^\pi \psi_2^* \frac{\partial}{\partial\theta}\left(s \frac{\partial\psi_1}{\partial\theta}\right) d\theta = \psi_2^* c\psi_1\big|_0^\pi - \tag{11}$$

$$\int_0^\pi \left[(\psi_2^*)' c\psi_1 - \psi_2^* s\psi_1\right] d\theta - \tag{12}$$

$$\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta \tag{13}$$

We can now integrate the last line by parts again to get rid of the derivatives of ψ_1:

$$-\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta = -\left(\psi_2^*\right)' s\psi_1\big|_0^\pi - \psi_2^* c\psi_1\big|_0^\pi + \tag{14}$$

$$\int_0^\pi \left[\psi_1 \left(\psi_2^*\right)'' s + \left(\psi_2^*\right)' c\psi_1\right] d\theta + \tag{15}$$

$$\int_0^\pi \left[\psi_1 \left(\psi_2^*\right)' c - \psi_2^* s\psi_1\right] d\theta \tag{16}$$

$$= -\psi_2^* c\psi_1\big|_0^\pi + \tag{17}$$

$$\int_0^\pi \left[\psi_1 \left(\psi_2^*\right)'' s + \left(\psi_2^*\right)' c\psi_1\right] d\theta + \tag{18}$$

$$\int_0^\pi \left[\psi_1 \left(\psi_2^*\right)' c - \psi_2^* s\psi_1\right] d\theta \tag{19}$$

Inserting this back into 11 and cancelling terms, we have

$$\int_0^\pi \psi_2^* \frac{\partial}{\partial\theta}\left(s \frac{\partial\psi_1}{\partial\theta}\right) d\theta = \int_0^\pi \left[\psi_1 \left(\psi_2^*\right)'' s + \left(\psi_2^*\right)' c\psi_1\right] d\theta \tag{20}$$

Comparing this with 7, we see that

$$\int_0^{\pi} \psi_2^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_1}{\partial\theta}\right) d\theta = \left[\int_0^{\pi} \psi_1^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_2}{\partial\theta}\right) d\theta\right]^* \tag{21}$$

Thus the first term in 1 is Hermitian. (As this first term involves no derivatives with respect to ϕ, the integration over ϕ is automatically Hermitian.)

For the second term in 1, we need to consider only the integral over ϕ, so we have

$$\int_0^{2\pi} \psi_2^* \frac{1}{\sin^2\theta}\frac{\partial^2\psi_1}{\partial\phi^2}\sin\theta\, d\phi = \frac{1}{s}\int_0^{2\pi} \psi_2^* \frac{\partial^2\psi_1}{\partial\phi^2}\, d\phi \tag{22}$$

(As we're integrating over ϕ, terms in θ act as constants and can be taken outside the integral.) The first integration by parts gives (where a prime now indicates a derivative with respect to ϕ):

$$\int_0^{2\pi} \psi_2^*\psi_1''\, d\phi = \psi_2^*\psi_1'\Big|_0^{2\pi} - \int_0^{2\pi} \left(\psi_2^*\right)'\psi_1'\, d\phi \tag{23}$$

$$= -\int_0^{2\pi} \left(\psi_2^*\right)'\psi_1'\, d\phi \tag{24}$$

This time, we're able to set the integrated term to zero, since $\phi = 0$ and $\phi = 2\pi$ refer to the same angle. A second integration by parts gives

$$-\int_0^{2\pi} \left(\psi_2^*\right)'\psi_1'\, d\phi = -\left(\psi_2^*\right)'\psi_1\Big|_0^{2\pi} + \int_0^{2\pi} \left(\psi_2^*\right)''\psi_1\, d\phi \tag{25}$$

$$= \int_0^{2\pi} \left(\psi_2^*\right)''\psi_1\, d\phi \tag{26}$$

$$= \left[\int_0^{2\pi} \psi_1^*\psi_2''\, d\phi\right]^* \tag{27}$$

Thus both terms in 1 are Hermitian, so the complete operator L^2 is also Hermitian.