

TOTAL ANGULAR MOMENTUM IS HERMITIAN

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.9.

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The total angular momentum operator L^2 can be written in spherical coordinates as

$$(1) \quad L^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

As L^2 is an observable, it should be Hermitian. We can verify this by showing that

$$(2) \quad \langle \psi_2 | L^2 | \psi_1 \rangle = \langle \psi_1 | L^2 | \psi_2 \rangle^*$$

In spherical coordinates, this becomes

$$(3) \quad \int \psi_2^* (L^2 \psi_1) d\Omega = \left[\int \psi_1^* (L^2 \psi_2) d\Omega \right]^*$$

The element of solid angle $d\Omega = \sin \theta d\theta d\phi$, so the full integral is

$$(4) \quad \int \psi_2^* (L^2 \psi_1) d\Omega = \int_0^{2\pi} \int_0^\pi \psi_2^* (L^2 \psi_1) \sin \theta d\theta d\phi$$

We can verify 3 by showing that it is true for each of the two terms in 1 separately. As usual for these sorts of integrals, we need to use integration by parts. To simplify things, we'll consider $-L^2/\hbar^2$ so we can deal only with the terms in the brackets in 1. We'll also use the shorthand notation

$$(5) \quad s \equiv \sin \theta$$

$$(6) \quad c \equiv \cos \theta$$

Also, a prime indicates a derivative with respect to θ : $\psi_1' \equiv \frac{\partial \psi_1}{\partial \theta}$, etc.
For the first term, we have, considering only the integration over θ :

$$\begin{aligned}
(7) \quad & \int_0^\pi \psi_2^* \frac{1}{s} \frac{\partial}{\partial \theta} \left(s \frac{\partial \psi_1}{\partial \theta} \right) s d\theta = \int_0^\pi [\psi_2^* c \psi_1' + \psi_2^* s \psi_1''] d\theta \\
(8) \quad & = \psi_2^* c \psi_1 |_0^\pi + \psi_2^* s \psi_1' |_0^\pi - \\
(9) \quad & \int_0^\pi [(\psi_2^*)' c \psi_1 - \psi_2^* s \psi_1'] d\theta - \\
(10) \quad & \int_0^\pi [(\psi_2^*)' s \psi_1' + \psi_2^* c \psi_1'] d\theta
\end{aligned}$$

The second term in 8 is zero since $\sin 0 = \sin \pi = 0$, but we can't ignore the first term, which is not, in general, zero. Thus we are left with

$$\begin{aligned}
(11) \quad & \int_0^\pi \psi_2^* \frac{\partial}{\partial \theta} \left(s \frac{\partial \psi_1}{\partial \theta} \right) d\theta = \psi_2^* c \psi_1 |_0^\pi - \\
(12) \quad & \int_0^\pi [(\psi_2^*)' c \psi_1 - \psi_2^* s \psi_1'] d\theta - \\
(13) \quad & \int_0^\pi [(\psi_2^*)' s \psi_1' + \psi_2^* c \psi_1'] d\theta
\end{aligned}$$

We can now integrate the last line by parts again to get rid of the derivatives of ψ_1 :

$$\begin{aligned}
(14) \quad & - \int_0^\pi [(\psi_2^*)' s \psi_1' + \psi_2^* c \psi_1'] d\theta = - (\psi_2^*)' s \psi_1 |_0^\pi - \psi_2^* c \psi_1 |_0^\pi + \\
(15) \quad & \int_0^\pi [\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c \psi_1] d\theta + \\
(16) \quad & \int_0^\pi [\psi_1 (\psi_2^*)' c - \psi_2^* s \psi_1'] d\theta \\
(17) \quad & = - \psi_2^* c \psi_1 |_0^\pi + \\
(18) \quad & \int_0^\pi [\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c \psi_1] d\theta + \\
(19) \quad & \int_0^\pi [\psi_1 (\psi_2^*)' c - \psi_2^* s \psi_1'] d\theta
\end{aligned}$$

Inserting this back into 11 and cancelling terms, we have

$$(20) \quad \int_0^\pi \psi_2^* \frac{\partial}{\partial \theta} \left(s \frac{\partial \psi_1}{\partial \theta} \right) d\theta = \int_0^\pi [\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c \psi_1] d\theta$$

Comparing this with 7, we see that

$$(21) \quad \int_0^\pi \psi_2^* \frac{\partial}{\partial \theta} \left(s \frac{\partial \psi_1}{\partial \theta} \right) d\theta = \left[\int_0^\pi \psi_1^* \frac{\partial}{\partial \theta} \left(s \frac{\partial \psi_2}{\partial \theta} \right) d\theta \right]^*$$

Thus the first term in 1 is Hermitian. (As this first term involves no derivatives with respect to ϕ , the integration over ϕ is automatically Hermitian.)

For the second term in 1, we need to consider only the integral over ϕ , so we have

$$(22) \quad \int_0^{2\pi} \psi_2^* \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi_1}{\partial \phi^2} \sin \theta d\phi = \frac{1}{s} \int_0^{2\pi} \psi_2^* \frac{\partial^2 \psi_1}{\partial \phi^2} d\phi$$

(As we're integrating over ϕ , terms in θ act as constants and can be taken outside the integral.) The first integration by parts gives (where a prime now indicates a derivative with respect to ϕ):

$$(23) \quad \int_0^{2\pi} \psi_2^* \psi_1'' d\phi = \psi_2^* \psi_1' \Big|_0^{2\pi} - \int_0^{2\pi} (\psi_2^*)' \psi_1' d\phi$$

$$(24) \quad = - \int_0^{2\pi} (\psi_2^*)' \psi_1' d\phi$$

This time, we're able to set the integrated term to zero, since $\phi = 0$ and $\phi = 2\pi$ refer to the same angle. A second integration by parts gives

$$(25) \quad - \int_0^{2\pi} (\psi_2^*)' \psi_1' d\phi = - (\psi_2^*)' \psi_1 \Big|_0^{2\pi} + \int_0^{2\pi} (\psi_2^*)'' \psi_1 d\phi$$

$$(26) \quad = \int_0^{2\pi} (\psi_2^*)'' \psi_1 d\phi$$

$$(27) \quad = \left[\int_0^{2\pi} \psi_1^* \psi_2'' d\phi \right]^*$$

Thus both terms in 1 are Hermitian, so the complete operator L^2 is also Hermitian.