TOTAL ANGULAR MOMENTUM IS HERMITIAN

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.9.

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The total angular momentum operator L^2 can be written in spherical coordinates as

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{0.1}$$

As L^2 is an observable, it should be Hermitian. We can verify this by showing that

$$\left\langle\psi_2\left|L^2\right|\psi_1\right\rangle=\left\langle\psi_1\left|L^2\right|\psi_2\right\rangle^* \tag{0.2}$$

In spherical coordinates, this becomes

$$\int\psi_2^*\left(L^2\psi_1\right)d\Omega=\left[\int\psi_1^*\left(L^2\psi_2\right)d\Omega\right]^* \tag{0.3}$$

The element of solid angle $d\Omega=\sin\theta\;d\theta\;d\phi$, so the full integral is

$$\int\psi_2^*\left(L^2\psi_1\right)d\Omega=\int_0^{2\pi}\int_0^{\pi}\psi_2^*\left(L^2\psi_1\right)\sin\theta\;d\theta\;d\phi \tag{0.4}$$

We can verify 0.3 by showing that it is true for each of the two terms in 0.1 separately. As usual for these sorts of integrals, we need to use integration by parts. To simplify things, we'll consider $-L^2/\hbar^2$ so we can deal only with the terms in the brackets in 0.1. We'll also use the shorthand notation

$$s \equiv \sin\theta \tag{0.5}$$
$$c \equiv \cos\theta \tag{0.6}$$

Also, a prime indicates a derivative with respect to θ: $\psi_1'\equiv\frac{\partial\psi_1}{\partial\theta}$, etc.

For the first term, we have, considering only the integration over θ:

$$\int_0^\pi \psi_2^* \frac{1}{s}\frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_1}{\partial\theta}\right) s\,d\theta = \int_0^\pi \left[\psi_2^* c\psi_1' + \psi_2^* s\psi_1''\right] d\theta \tag{0.7}$$

$$= \psi_2^* c\psi_1\big|_0^\pi + \psi_2^* s\psi_1'\big|_0^\pi - \tag{0.8}$$

$$\int_0^\pi \left[(\psi_2^*)' c\psi_1 - \psi_2^* s\psi_1\right] d\theta - \tag{0.9}$$

$$\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta \tag{0.10}$$

The second term in 0.8 is zero since $\sin 0 = \sin\pi = 0$, but we can't ignore the first term, which is not, in general, zero. Thus we are left with

$$\int_0^\pi \psi_2^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_1}{\partial\theta}\right) d\theta = \psi_2^* c\psi_1\big|_0^\pi - \tag{0.11}$$

$$\int_0^\pi \left[(\psi_2^*)' c\psi_1 - \psi_2^* s\psi_1\right] d\theta - \tag{0.12}$$

$$\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta \tag{0.13}$$

We can now integrate the last line by parts again to get rid of the derivatives of ψ_1:

$$-\int_0^\pi \left[(\psi_2^*)' s\psi_1' + \psi_2^* c\psi_1'\right] d\theta = -\,(\psi_2^*)' s\psi_1\big|_0^\pi - \psi_2^* c\psi_1\big|_0^\pi + \tag{0.14}$$

$$\int_0^\pi \left[\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c\psi_1\right] d\theta + \tag{0.15}$$

$$\int_0^\pi \left[\psi_1 (\psi_2^*)' c - \psi_2^* s\psi_1\right] d\theta \tag{0.16}$$

$$= -\,\psi_2^* c\psi_1\big|_0^\pi + \tag{0.17}$$

$$\int_0^\pi \left[\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c\psi_1\right] d\theta + \tag{0.18}$$

$$\int_0^\pi \left[\psi_1 (\psi_2^*)' c - \psi_2^* s\psi_1\right] d\theta \tag{0.19}$$

Inserting this back into 0.11 and cancelling terms, we have

$$\int_0^\pi \psi_2^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_1}{\partial\theta}\right) d\theta = \int_0^\pi \left[\psi_1 (\psi_2^*)'' s + (\psi_2^*)' c\psi_1\right] d\theta \tag{0.20}$$

Comparing this with 0.7, we see that

$$\int_0^{\pi} \psi_2^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_1}{\partial\theta}\right) d\theta = \left[\int_0^{\pi} \psi_1^* \frac{\partial}{\partial\theta}\left(s\frac{\partial\psi_2}{\partial\theta}\right) d\theta\right]^* \tag{0.21}$$

Thus the first term in 0.1 is Hermitian. (As this first term involves no derivatives with respect to ϕ, the integration over ϕ is automatically Hermitian.)

For the second term in 0.1, we need to consider only the integral over ϕ, so we have

$$\int_0^{2\pi} \psi_2^* \frac{1}{\sin^2\theta}\frac{\partial^2\psi_1}{\partial\phi^2}\sin\theta\, d\phi = \frac{1}{s}\int_0^{2\pi} \psi_2^* \frac{\partial^2\psi_1}{\partial\phi^2}\, d\phi \tag{0.22}$$

(As we're integrating over ϕ, terms in θ act as constants and can be taken outside the integral.) The first integration by parts gives (where a prime now indicates a derivative with respect to ϕ):

$$\int_0^{2\pi} \psi_2^*\psi_1''\, d\phi = \psi_2^*\psi_1'\Big|_0^{2\pi} - \int_0^{2\pi} (\psi_2^*)'\,\psi_1'\, d\phi \tag{0.23}$$

$$= -\int_0^{2\pi} (\psi_2^*)'\,\psi_1'\, d\phi \tag{0.24}$$

This time, we're able to set the integrated term to zero, since $\phi = 0$ and $\phi = 2\pi$ refer to the same angle. A second integration by parts gives

$$-\int_0^{2\pi} (\psi_2^*)'\,\psi_1'\, d\phi = -\left(\psi_2^*\right)'\psi_1\Big|_0^{2\pi} + \int_0^{2\pi} (\psi_2^*)''\,\psi_1\, d\phi \tag{0.25}$$

$$= \int_0^{2\pi} (\psi_2^*)''\,\psi_1\, d\phi \tag{0.26}$$

$$= \left[\int_0^{2\pi} \psi_1^*\psi_2''\, d\phi\right]^* \tag{0.27}$$

Thus both terms in 0.1 are Hermitian, so the complete operator L^2 is also Hermitian.