SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES FOR M=0

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.10.

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The total angular momentum operator L^2 can be written in spherical coordinates as

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{1}$$

Since $\left[L^2,L_z\right]=0$, we can find a basis consisting of simultaneous eigenfunctions of L^2 and L_z. Suppose we call these states $|\alpha\beta\rangle$, where α is the eigenvalue of L^2 and β is the eigenvalue of L_z. In spherical coordinates, we know that

$$L_z=-i\hbar\frac{\partial}{\partial\phi} \tag{2}$$

and that its eigenvalues are $m\hbar$ for integer values of m. Thus we can separate the θ and ϕ dependence in the eigenstates and write

$$\psi_{\alpha m}(\theta,\phi)=P_\alpha^m(\theta)e^{im\phi} \tag{3}$$

We therefore have the eigenvalue equation

$$\begin{aligned}L^2|\alpha m\rangle &= \alpha|\alpha m\rangle &(4)\\ L^2\psi_{\alpha m}(\theta,\phi) &= \alpha\psi_{\alpha m}(\theta,\phi) &(5)\end{aligned}$$

Combining 3 with 1, we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_{\alpha m}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi_{\alpha m}}{\partial\phi^2}=\alpha\psi_{\alpha m} \tag{6}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\alpha^m}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}P_\alpha^m+\frac{\alpha}{\hbar^2}P_\alpha^m=0 \tag{7}$$

We'd like to show that solutions of this equation require that (1)

$$\begin{aligned} \alpha &= \hbar^2\ell(\ell+1) & (8) \\ |m| &\leq \ell & (9) \end{aligned}$$

for $\ell = 0, 1, 2, \ldots$. In the problem given in Shankar, we tackle the less demanding case of $m = 0$ and demonstrate only the result for α. We begin by transforming 7 using the variable substitution:

$$u \equiv \cos\theta \tag{10}$$

This gives us

$$du = -\sin\theta\, d\theta \tag{11}$$

so that 7 becomes

$$\begin{aligned} \frac{-\sin\theta}{\sin\theta}\frac{\partial}{\partial u}\left(-\sin^2\theta\frac{\partial P_\alpha^0}{\partial u}\right)+\frac{\alpha}{\hbar^2}P_\alpha^0 = 0 & \quad (12) \\ \frac{\partial}{\partial u}\left(\left(1-u^2\right)\frac{\partial P_\alpha^0}{\partial u}\right)+\frac{\alpha}{\hbar^2}P_\alpha^0 = 0 & \quad (13) \\ \left(1-u^2\right)\frac{\partial^2 P_\alpha^0}{\partial u^2}-2u\frac{\partial P_\alpha^0}{\partial u}+\frac{\alpha}{\hbar^2}P_\alpha^0 = 0 & \quad (14) \end{aligned}$$

We can use a power series to solve this by defining

$$\begin{aligned} P_\alpha^0(u) &= \sum_{n=0}^{\infty} C_n u^n & (15) \\ \frac{\partial P_\alpha^0}{\partial u} &= \sum_{n=0}^{\infty} C_n n u^{n-1} & (16) \\ \frac{\partial^2 P_\alpha^0}{\partial u^2} &= \sum_{n=0}^{\infty} C_n n(n-1)u^{n-2} & (17) \\ &= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)u^n & (18) \end{aligned}$$

Plugging these into 14 and collecting terms, we get

$$P_\alpha^0(u) = \sum_{n=0}^{\infty}\left[C_{n+2}(n+2)(n+1)+C_n\left(-n(n-1)-2n+\frac{\alpha}{\hbar^2}\right)\right]u^n = 0 \tag{19}$$

If a power series equals zero, the coefficient of each power of u must be zero (power series theorem from math), so we get the recurrence relation

$$C_{n+2} = \frac{n(n-1)+2n-\frac{\alpha}{\hbar^2}}{(n+2)(n+1)}C_n \tag{20}$$

$$= \frac{n^2+n-\frac{\alpha}{\hbar^2}}{n^2+3n+2}C_n \tag{21}$$

For large n we have

$$C_{n+2} \to \frac{n^2}{n^2}C_n = C_n \tag{22}$$

Since $u = \cos\theta$, $u \in [-1,1]$ and the series must converge for all these values. Although the power series $\sum_{n=0}^{\infty} u^n$ converges if $|u| < 1$ (that's the standard geometric series), it clearly diverges if $u = 1$. Thus we require the series to terminate, which imposes a condition on α:

$$\alpha = \ell(\ell+1)\hbar^2 \tag{23}$$

for some integer value $\ell = 0,1,2,\ldots$. Since choosing a value for ℓ can be done only once in any given series, and the recursion relation relates every *second* coefficient, this implies that either all even coefficients or all odd coefficients must be zero. Thus $P_\alpha^0(u)$ is either a sum of even powers (making it an even function) or of odd powers (making it an odd function) only.

The first few values of $P_\alpha^0(u)$ are found by choosing values for C_0 and C_1 and then generating all higher coefficients using 21. If we take

$$C_0 = 1 \tag{24}$$

$$C_1 = 0 \tag{25}$$

then if we choose $\ell = 0$ we get

$$P_0^0 = 1 \tag{26}$$

Taking

$$C_0 = 0 \tag{27}$$

$$C_1 = 1 \tag{28}$$

and $\ell = 1$ gives

$$P_1^0 = u = \cos\theta \tag{29}$$

Reverting to an even series and taking $\ell=2$ we have from 21

$$C_2 = -\frac{\alpha}{2\hbar^2}C_0 = -\frac{\ell(\ell+1)}{2}(1) = -3 \tag{30}$$

$$P_2^0 = 1-3u^2 = 1-3\cos^2\theta \tag{31}$$

These values for P_ℓ^0 agree with the spherical harmonics Y_ℓ^0 apart from the constant scaling factors in each case. See Shankar's equation 12.5.39 for comparison.

PINGBACKS

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