SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES FOR M=0

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.10.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The total angular momentum operator L^2 can be written in spherical coordinates as

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \tag{1}$$

Since $\left[L^2, L_z\right] = 0$, we can find a basis consisting of simultaneous eigenfunctions of L^2 and L_z. Suppose we call these states $|\alpha\beta\rangle$, where α is the eigenvalue of L^2 and β is the eigenvalue of L_z. In spherical coordinates, we know that

$$L_z = -i\hbar \frac{\partial}{\partial\phi} \tag{2}$$

and that its eigenvalues are $m\hbar$ for integer values of m. Thus we can separate the θ and ϕ dependence in the eigenstates and write

$$\psi_{\alpha m}(\theta,\phi) = P_\alpha^m(\theta)\, e^{im\phi} \tag{3}$$

We therefore have the eigenvalue equation

$$\begin{aligned} L^2 |\alpha m\rangle &= \alpha |\alpha m\rangle &(4)\\ L^2 \psi_{\alpha m}(\theta,\phi) &= \alpha \psi_{\alpha m}(\theta,\phi) &(5) \end{aligned}$$

Combining 3 with 1, we have

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial \psi_{\alpha m}}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi_{\alpha m}}{\partial\phi^2} = \alpha \psi_{\alpha m} \tag{6}$$

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial P_\alpha^m}{\partial\theta} \right) - \frac{m^2}{\sin^2\theta} P_\alpha^m + \frac{\alpha}{\hbar^2} P_\alpha^m = 0 \tag{7}$$

We'd like to show that solutions of this equation require that (1)

$$\alpha = \hbar^2\ell(\ell+1) \tag{8}$$
$$|m| \leq \ell \tag{9}$$

for $\ell = 0, 1, 2, \ldots$. In the problem given in Shankar, we tackle the less demanding case of $m = 0$ and demonstrate only the result for α. We begin by transforming 7 using the variable substitution:

$$u \equiv \cos\theta \tag{10}$$

This gives us

$$du = -\sin\theta\, d\theta \tag{11}$$

so that 7 becomes

$$\frac{-\sin\theta}{\sin\theta}\frac{\partial}{\partial u}\left(-\sin^2\theta\frac{\partial P_\alpha^0}{\partial u}\right) + \frac{\alpha}{\hbar^2}P_\alpha^0 = 0 \tag{12}$$
$$\frac{\partial}{\partial u}\left(\left(1-u^2\right)\frac{\partial P_\alpha^0}{\partial u}\right) + \frac{\alpha}{\hbar^2}P_\alpha^0 = 0 \tag{13}$$
$$\left(1-u^2\right)\frac{\partial^2 P_\alpha^0}{\partial u^2} - 2u\frac{\partial P_\alpha^0}{\partial u} + \frac{\alpha}{\hbar^2}P_\alpha^0 = 0 \tag{14}$$

We can use a power series to solve this by defining

$$P_\alpha^0(u) = \sum_{n=0}^{\infty} C_n u^n \tag{15}$$
$$\frac{\partial P_\alpha^0}{\partial u} = \sum_{n=0}^{\infty} C_n n u^{n-1} \tag{16}$$
$$\frac{\partial^2 P_\alpha^0}{\partial u^2} = \sum_{n=0}^{\infty} C_n n(n-1)u^{n-2} \tag{17}$$
$$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)u^n \tag{18}$$

Plugging these into 14 and collecting terms, we get

$$P_\alpha^0(u) = \sum_{n=0}^{\infty}\left[C_{n+2}(n+2)(n+1) + C_n\left(-n(n-1) - 2n + \frac{\alpha}{\hbar^2}\right)\right]u^n = 0 \tag{19}$$

If a power series equals zero, the coefficient of each power of u must be zero (power series theorem from math), so we get the recurrence relation

$$C_{n+2} = \frac{n(n-1)+2n-\frac{\alpha}{\hbar^2}}{(n+2)(n+1)}C_n \tag{20}$$

$$= \frac{n^2+n-\frac{\alpha}{\hbar^2}}{n^2+3n+2}C_n \tag{21}$$

For large n we have

$$C_{n+2} \to \frac{n^2}{n^2}C_n = C_n \tag{22}$$

Since $u = \cos\theta$, $u \in [-1,1]$ and the series must converge for all these values. Although the power series $\sum_{n=0}^{\infty} u^n$ converges if $|u| < 1$ (that's the standard geometric series), it clearly diverges if $u = 1$. Thus we require the series to terminate, which imposes a condition on α:

$$\alpha = \ell(\ell+1)\hbar^2 \tag{23}$$

for some integer value $\ell = 0,1,2,\ldots$. Since choosing a value for ℓ can be done only once in any given series, and the recursion relation relates every *second* coefficient, this implies that either all even coefficients or all odd coefficients must be zero. Thus $P_\alpha^0(u)$ is either a sum of even powers (making it an even function) or of odd powers (making it an odd function) only.

The first few values of $P_\alpha^0(u)$ are found by choosing values for C_0 and C_1 and then generating all higher coefficients using 21. If we take

$$C_0 = 1 \tag{24}$$

$$C_1 = 0 \tag{25}$$

then if we choose $\ell = 0$ we get

$$P_0^0 = 1 \tag{26}$$

Taking

$$C_0 = 0 \tag{27}$$

$$C_1 = 1 \tag{28}$$

and $\ell = 1$ gives

$$P_1^0 = u = \cos\theta \tag{29}$$

Reverting to an even series and taking $\ell=2$ we have from 21

$$C_2 = -\frac{\alpha}{2\hbar^2}C_0 = -\frac{\ell(\ell+1)}{2}(1) = -3 \tag{30}$$

$$P_2^0 = 1-3u^2 = 1-3\cos^2\theta \tag{31}$$

These values for P_ℓ^0 agree with the spherical harmonics Y_ℓ^0 apart from the constant scaling factors in each case. See Shankar's equation 12.5.39 for comparison.

PINGBACKS

Pingback: angular momentum and parity