

SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES FOR M=0

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.10.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The total angular momentum operator L^2 can be written in spherical coordinates as

$$(0.1) \quad L^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

Since $[L^2, L_z] = 0$, we can find a basis consisting of simultaneous eigenfunctions of L^2 and L_z . Suppose we call these states $|\alpha\beta\rangle$, where α is the eigenvalue of L^2 and β is the eigenvalue of L_z . In spherical coordinates, we know that

$$(0.2) \quad L_z = -i\hbar \frac{\partial}{\partial \phi}$$

and that its eigenvalues are $m\hbar$ for integer values of m . Thus we can separate the θ and ϕ dependence in the eigenstates and write

$$(0.3) \quad \psi_{\alpha m}(\theta, \phi) = P_{\alpha}^m(\theta) e^{im\phi}$$

We therefore have the eigenvalue equation

$$(0.4) \quad L^2 |\alpha m\rangle = \alpha |\alpha m\rangle$$

$$(0.5) \quad L^2 \psi_{\alpha m}(\theta, \phi) = \alpha \psi_{\alpha m}(\theta, \phi)$$

Combining 0.3 with 0.1, we have

$$(0.6) \quad \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi_{\alpha m}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Psi_{\alpha m}}{\partial \phi^2} = \alpha \Psi_{\alpha m}$$

$$(0.7) \quad \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P_{\alpha}^m}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} P_{\alpha}^m + \frac{\alpha}{\hbar^2} P_{\alpha}^m = 0$$

We'd like to show that solutions of this equation require that (1)

$$(0.8) \quad \alpha = \hbar^2 \ell(\ell + 1)$$

$$(0.9) \quad |m| \leq \ell$$

for $\ell = 0, 1, 2, \dots$. In the problem given in Shankar, we tackle the less demanding case of $m = 0$ and demonstrate only the result for α . We begin by transforming 0.7 using the variable substitution:

$$(0.10) \quad u \equiv \cos \theta$$

This gives us

$$(0.11) \quad du = -\sin \theta d\theta$$

so that 0.7 becomes

$$(0.12) \quad \frac{-\sin \theta}{\sin \theta} \frac{\partial}{\partial u} \left(-\sin^2 \theta \frac{\partial P_{\alpha}^0}{\partial u} \right) + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0$$

$$(0.13) \quad \frac{\partial}{\partial u} \left((1 - u^2) \frac{\partial P_{\alpha}^0}{\partial u} \right) + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0$$

$$(0.14) \quad (1 - u^2) \frac{\partial^2 P_{\alpha}^0}{\partial u^2} - 2u \frac{\partial P_{\alpha}^0}{\partial u} + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0$$

We can use a power series to solve this by defining

$$(0.15) \quad P_{\alpha}^0(u) = \sum_{n=0}^{\infty} C_n u^n$$

$$(0.16) \quad \frac{\partial P_{\alpha}^0}{\partial u} = \sum_{n=0}^{\infty} C_n n u^{n-1}$$

$$(0.17) \quad \frac{\partial^2 P_{\alpha}^0}{\partial u^2} = \sum_{n=0}^{\infty} C_n n(n-1) u^{n-2}$$

$$(0.18) \quad = \sum_{n=0}^{\infty} C_{n+2} (n+2)(n+1) u^n$$

Plugging these into 0.14 and collecting terms, we get

$$(0.19) \quad P_{\alpha}^0(u) = \sum_{n=0}^{\infty} \left[C_{n+2} (n+2)(n+1) + C_n \left(-n(n-1) - 2n + \frac{\alpha}{\hbar^2} \right) \right] u^n = 0$$

If a power series equals zero, the coefficient of each power of u must be zero (power series theorem from math), so we get the recurrence relation

$$(0.20) \quad C_{n+2} = \frac{n(n-1) + 2n - \frac{\alpha}{\hbar^2}}{(n+2)(n+1)} C_n$$

$$(0.21) \quad = \frac{n^2 + n - \frac{\alpha}{\hbar^2}}{n^2 + 3n + 2} C_n$$

For large n we have

$$(0.22) \quad C_{n+2} \rightarrow \frac{n^2}{n^2} C_n = C_n$$

Since $u = \cos \theta$, $u \in [-1, 1]$ and the series must converge for all these values. Although the power series $\sum_{n=0}^{\infty} u^n$ converges if $|u| < 1$ (that's the standard geometric series), it clearly diverges if $u = 1$. Thus we require the series to terminate, which imposes a condition on α :

$$(0.23) \quad \alpha = \ell(\ell+1)\hbar^2$$

for some integer value $\ell = 0, 1, 2, \dots$. Since choosing a value for ℓ can be done only once in any given series, and the recursion relation relates every *second* coefficient, this implies that either all even coefficients or all odd coefficients must be zero. Thus $P_{\alpha}^0(u)$ is either a sum of even powers (making it an even function) or of odd powers (making it an odd function) only.

The first few values of $P_{\alpha}^0(u)$ are found by choosing values for C_0 and C_1 and then generating all higher coefficients using 0.21. If we take

$$(0.24) \quad C_0 = 1$$

$$(0.25) \quad C_1 = 0$$

then if we choose $\ell = 0$ we get

$$(0.26) \quad P_0^0 = 1$$

Taking

$$(0.27) \quad C_0 = 0$$

$$(0.28) \quad C_1 = 1$$

and $\ell = 1$ gives

$$(0.29) \quad P_1^0 = u = \cos \theta$$

Reverting to an even series and taking $\ell = 2$ we have from 0.21

$$(0.30) \quad C_2 = -\frac{\alpha}{2\hbar^2} C_0 = -\frac{\ell(\ell+1)}{2} (1) = -3$$

$$(0.31) \quad P_2^0 = 1 - 3u^2 = 1 - 3\cos^2 \theta$$

These values for P_ℓ^0 agree with the spherical harmonics Y_ℓ^0 apart from the constant scaling factors in each case. See Shankar's equation 12.5.39 for comparison.

PINGBACKS

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