SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES FOR M=0

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.10.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The total angular momentum operator L^2 can be written in spherical coordinates as

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{1}$$

Since $\left[L^2, L_z\right] = 0$, we can find a basis consisting of simultaneous eigenfunctions of L^2 and L_z. Suppose we call these states $|\alpha\beta\rangle$, where α is the eigenvalue of L^2 and β is the eigenvalue of L_z. In spherical coordinates, we know that

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{2}$$

and that its eigenvalues are $m\hbar$ for integer values of m. Thus we can separate the θ and ϕ dependence in the eigenstates and write

$$\psi_{\alpha m}(\theta,\phi) = P_\alpha^m(\theta)\, e^{im\phi} \tag{3}$$

We therefore have the eigenvalue equation

$$L^2|\alpha m\rangle = \alpha|\alpha m\rangle \tag{4}$$

$$L^2\psi_{\alpha m}(\theta,\phi) = \alpha\psi_{\alpha m}(\theta,\phi) \tag{5}$$

Combining 3 with 1, we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_{\alpha m}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi_{\alpha m}}{\partial\phi^2}=\alpha\psi_{\alpha m} \tag{6}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\alpha^m}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}P_\alpha^m+\frac{\alpha}{\hbar^2}P_\alpha^m=0 \tag{7}$$

We'd like to show that solutions of this equation require that (1)

$$\alpha = \hbar^2\ell(\ell+1) \tag{8}$$

$$|m| \leq \ell \tag{9}$$

for $\ell = 0, 1, 2, \ldots$. In the problem given in Shankar, we tackle the less demanding case of $m = 0$ and demonstrate only the result for α. We begin by transforming 7 using the variable substitution:

$$u \equiv \cos\theta \tag{10}$$

This gives us

$$du = -\sin\theta\, d\theta \tag{11}$$

so that 7 becomes

$$\frac{-\sin\theta}{\sin\theta}\frac{\partial}{\partial u}\left(-\sin^2\theta\frac{\partial P_\alpha^0}{\partial u}\right)+\frac{\alpha}{\hbar^2}P_\alpha^0=0 \tag{12}$$

$$\frac{\partial}{\partial u}\left(\left(1-u^2\right)\frac{\partial P_\alpha^0}{\partial u}\right)+\frac{\alpha}{\hbar^2}P_\alpha^0=0 \tag{13}$$

$$\left(1-u^2\right)\frac{\partial^2 P_\alpha^0}{\partial u^2}-2u\frac{\partial P_\alpha^0}{\partial u}+\frac{\alpha}{\hbar^2}P_\alpha^0=0 \tag{14}$$

We can use a power series to solve this by defining

$$P_\alpha^0(u) = \sum_{n=0}^{\infty} C_n u^n \tag{15}$$

$$\frac{\partial P_\alpha^0}{\partial u} = \sum_{n=0}^{\infty} C_n n u^{n-1} \tag{16}$$

$$\frac{\partial^2 P_\alpha^0}{\partial u^2} = \sum_{n=0}^{\infty} C_n n(n-1)u^{n-2} \tag{17}$$

$$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)u^n \tag{18}$$

Plugging these into 14 and collecting terms, we get

(19)
$$P_{\alpha}^{0}(u)=\sum_{n=0}^{\infty}\left[C_{n+2}(n+2)(n+1)+C_{n}\left(-n(n-1)-2n+\frac{\alpha}{\hbar^{2}}\right)\right]u^{n}=0$$

If a power series equals zero, the coefficient of each power of u must be zero (power series theorem from math), so we get the recurrence relation

$$C_{n+2} = \frac{n(n-1)+2n-\frac{\alpha}{\hbar^{2}}}{(n+2)(n+1)}C_{n} \tag{20}$$

$$= \frac{n^{2}+n-\frac{\alpha}{\hbar^{2}}}{n^{2}+3n+2}C_{n} \tag{21}$$

For large n we have

$$C_{n+2}\rightarrow\frac{n^{2}}{n^{2}}C_{n}=C_{n} \tag{22}$$

Since $u=\cos\theta$, $u\in[-1,1]$ and the series must converge for all these values. Although the power series $\sum_{n=0}^{\infty}u^{n}$ converges if $|u|<1$ (that's the standard geometric series), it clearly diverges if $u=1$. Thus we require the series to terminate, which imposes a condition on α:

$$\alpha=\ell(\ell+1)\hbar^{2} \tag{23}$$

for some integer value $\ell=0,1,2,\ldots$. Since choosing a value for ℓ can be done only once in any given series, and the recursion relation relates every *second* coefficient, this implies that either all even coefficients or all odd coefficients must be zero. Thus $P_{\alpha}^{0}(u)$ is either a sum of even powers (making it an even function) or of odd powers (making it an odd function) only.

The first few values of $P_{\alpha}^{0}(u)$ are found by choosing values for C_0 and C_1 and then generating all higher coefficients using 21. If we take

$$C_{0} = 1 \tag{24}$$

$$C_{1} = 0 \tag{25}$$

then if we choose $\ell=0$ we get

$$P_{0}^{0}=1 \tag{26}$$

Taking

$$C_0 = 0 \tag{27}$$
$$C_1 = 1 \tag{28}$$

and $\ell = 1$ gives

$$P_1^0 = u = \cos\theta \tag{29}$$

Reverting to an even series and taking $\ell = 2$ we have from 21

$$C_2 = -\frac{\alpha}{2\hbar^2}C_0 = -\frac{\ell(\ell+1)}{2}(1) = -3 \tag{30}$$
$$P_2^0 = 1 - 3u^2 = 1 - 3\cos^2\theta \tag{31}$$

These values for P_ℓ^0 agree with the spherical harmonics Y_ℓ^0 apart from the constant scaling factors in each case. See Shankar's equation 12.5.39 for comparison.

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