

SPHERICAL HARMONICS USING THE LOWERING OPERATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The raising and lowering operators for angular momentum are

$$L_{\pm} \equiv L_x \pm iL_y \quad (1)$$

On a state $|\ell m\rangle$ in the basis of eigenstates of L^2 and L_z , they have the effect:

$$L_{\pm} |\ell m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle \quad (2)$$

This means that, if we can find the top state $|\ell \ell\rangle$, we can find the state for all lower values of m by applying L_- successively. To illustrate the process we'll derive the 3 states for $\ell = 1$. The top state $|11\rangle$ can be obtained by following the derivation given in Shankar from his equation 12.5.28 onwards. In spherical coordinates, the raising and lowering operators have the form

$$L_{\pm} = \pm \hbar e^{\pm i\phi} \left[\frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \phi} \right] \quad (3)$$

Applying L_+ to the top state $|11\rangle$ must give zero, so if ψ_1^1 is the representation of this state in spherical coordinates, we must solve the differential equation

$$\left[\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right] \psi_1^1 = 0 \quad (4)$$

Since ψ_1^1 is also an eigenfunction of L_z with eigenvalue \hbar , we know that

$$\psi_1^1 = U_1^1(r, \theta) e^{i\phi} \quad (5)$$

Thus 4 becomes

$$\left(\frac{\partial}{\partial \theta} - \cot \theta \right) U_1^1 = 0 \quad (6)$$

This can be solved by writing it in the form

$$\frac{dU_1^1}{U_1^1} = \frac{d(\sin\theta)}{\sin\theta} \quad (7)$$

$$\ln U_1^1 = \ln(\sin\theta) + \ln R(r) + \ln A \quad (8)$$

where R is some unspecified function of r , and A is a constant. We therefore have

$$U_1^1(r, \theta) = R(r)(A \sin\theta) \quad (9)$$

If we ignore R for now, we can normalize over the angular coordinates by requiring

$$\int |A \sin\theta|^2 d\Omega = 1 \quad (10)$$

The element $d\Omega$ of solid angle is

$$d\Omega = \sin\theta d\phi d\theta \quad (11)$$

so we have

$$|A|^2 \int_0^\pi \int_0^{2\pi} \sin^3\theta d\phi d\theta = 2\pi |A|^2 \int_0^\pi \sin\theta (1 - \cos^2\theta) d\theta \quad (12)$$

$$= \frac{8\pi}{3} |A|^2 \quad (13)$$

$$A = \sqrt{\frac{3}{8\pi}} \quad (14)$$

Thus the spherical harmonic Y_1^1 is (using Shankar's normalization convention of multiplying by $(-1)^\ell$):

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \quad (15)$$

We can now get Y_1^0 by applying L_- to Y_1^1 . From 2 we have

$$L_- Y_1^1 = \hbar \sqrt{(1+1)(1-1+1)} Y_1^0 \quad (16)$$

$$= \sqrt{2} \hbar Y_1^0 \quad (17)$$

From 3 we have

$$L_- Y_1^1 = -\hbar e^{-i\phi} \left[\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^1 \quad (18)$$

$$= -\hbar e^{-i\phi} \left(-\sqrt{\frac{3}{8\pi}} \right) [\cos \theta - i \cot \theta (i \sin \theta)] e^{i\phi} \quad (19)$$

$$= 2\hbar \sqrt{\frac{3}{8\pi}} \cos \theta \quad (20)$$

Comparing the last two results gives

$$\sqrt{2}\hbar Y_1^0 = 2\sqrt{\frac{3}{8\pi}} \cos \theta \quad (21)$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \quad (22)$$

Repeating the process, we get

$$L_- Y_1^0 = \hbar \sqrt{(1+0)(1-0+1)} Y_1^{-1} \quad (23)$$

$$= \sqrt{2}\hbar Y_1^{-1} \quad (24)$$

Also

$$L_- Y_1^0 = -\hbar e^{-i\phi} \left[\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^0 \quad (25)$$

$$= -\hbar e^{-i\phi} \sqrt{\frac{3}{4\pi}} (-\sin \theta - 0) \quad (26)$$

$$= \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi} \quad (27)$$

Thus

$$\sqrt{2}\hbar Y_1^{-1} = \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi} \quad (28)$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi} \quad (29)$$

Comparing these results with Shankar's equation 12.5.39 we see that they match. [This exercise is similar to one we did earlier, where we used the raising operator to generate spherical harmonics with higher values of m .]

PINGBACKS

Pingback: angular momentum and parity