SPHERICAL HARMONICS USING THE LOWERING OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The raising and lowering operators for angular momentum are

$$L_{\pm}\equiv L_{x}\pm iL_{y} \tag{1}$$

On a state $|\ell m\rangle$ in the basis of eigenstates of L^{2} and L_{z}, they have the effect:

$$L_{\pm}|\ell m\rangle=\hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}\,|\ell,m\pm1\rangle \tag{2}$$

This means that, if we can find the top state $|\ell\ell\rangle$, we can find the state for all lower values of m by applying L_{-} successively. To illustrate the process we'll derive the 3 states for $\ell=1$. The top state $|11\rangle$ can be obtained by following the derivation given in Shankar from his equation 12.5.28 onwards. In spherical coordinates, the raising and lowering operators have the form

$$L_{\pm}=\pm\hbar e^{\pm i\phi}\left[\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\phi}\right] \tag{3}$$

Applying L_{+} to the top state $|11\rangle$ must give zero, so if ψ_{1}^{1} is the representation of this state in spherical coordinates, we must solve the differential equation

$$\left[\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right]\psi_{1}^{1}=0 \tag{4}$$

Since ψ_{1}^{1} is also an eigenfunction of L_{z} with eigenvalue $\hbar$, we know that

$$\psi_{1}^{1}=U_{1}^{1}(r,\theta)\,e^{i\phi} \tag{5}$$

Thus 4 becomes

$$\left(\frac{\partial}{\partial\theta}-\cot\theta\right)U_{1}^{1}=0 \tag{6}$$

This can be solved by writing it in the form

$$\frac{dU_1^1}{U_1^1} = \frac{d(\sin\theta)}{\sin\theta} \tag{7}$$

$$\ln U_1^1 = \ln(\sin\theta) + \ln R(r) + \ln A \tag{8}$$

where R is some unspecified function of r, and A is a constant. We therefore have

$$U_1^1(r,\theta) = R(r)(A\sin\theta) \tag{9}$$

If we ignore R for now, we can normalize over the angular coordinates by requiring

$$\int |A\sin\theta|^2 d\Omega = 1 \tag{10}$$

The element $d\Omega$ of solid angle is

$$d\Omega = \sin\theta\, d\phi\, d\theta \tag{11}$$

so we have

$$|A|^2 \int_0^{\pi}\int_0^{2\pi} \sin^3\theta\, d\phi\, d\theta = 2\pi|A|^2 \int_0^{\pi} \sin\theta\left(1-\cos^2\theta\right)d\theta \tag{12}$$

$$= \frac{8\pi}{3}|A|^2 \tag{13}$$

$$A = \sqrt{\frac{3}{8\pi}} \tag{14}$$

Thus the spherical harmonic Y_1^1 is (using Shankar's normalization convention of multiplying by $(-1)^{\ell}$):

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \tag{15}$$

We can now get Y_1^0 by applying L_- to Y_1^1. From 2 we have

$$L_-Y_1^1 = \hbar\sqrt{(1+1)(1-1+1)}Y_1^0 \tag{16}$$

$$= \sqrt{2}\hbar Y_1^0 \tag{17}$$

From 3 we have

$$L_-Y_1^1 = -\hbar e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right]Y_1^1 \tag{18}$$

$$= -\hbar e^{-i\phi}\left(-\sqrt{\frac{3}{8\pi}}\right)\left[\cos\theta - i\cot\theta\left(i\sin\theta\right)\right]e^{i\phi} \tag{19}$$

$$= 2\hbar\sqrt{\frac{3}{8\pi}}\cos\theta \tag{20}$$

Comparing the last two results gives

$$\sqrt{2}\hbar Y_1^0 = 2\sqrt{\frac{3}{8\pi}}\cos\theta \tag{21}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{22}$$

Repeating the process, we get

$$L_-Y_1^0 = \hbar\sqrt{(1+0)(1-0+1)}Y_1^{-1} \tag{23}$$

$$= \sqrt{2}\hbar Y_1^{-1} \tag{24}$$

Also

$$L_-Y_1^0 = -\hbar e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right]Y_1^0 \tag{25}$$

$$= -\hbar e^{-i\phi}\sqrt{\frac{3}{4\pi}}\left(-\sin\theta - 0\right) \tag{26}$$

$$= \hbar\sqrt{\frac{3}{4\pi}}\sin\theta e^{-i\phi} \tag{27}$$

Thus

$$\sqrt{2}\hbar Y_1^{-1} = \hbar\sqrt{\frac{3}{4\pi}}\sin\theta e^{-i\phi} \tag{28}$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \tag{29}$$

Comparing these results with Shankar's equation 12.5.39 we see that they match. [This exercise is similar to one we did earlier, where we used the raising operator to generate spherical harmonics with higher values of m.]

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