SPHERICAL HARMONICS USING THE LOWERING OPERATOR

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The raising and lowering operators for angular momentum are

\[
L_\pm \equiv L_x \pm iL_y \tag{1}
\]

On a state \(|\ell m\rangle\) in the basis of eigenstates of \(L^2\) and \(L_z\), they have the effect:

\[
L_\pm |\ell m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle \tag{2}
\]

This means that, if we can find the top state \(|\ell \ell\rangle\), we can find the state for all lower values of \(m\) by applying \(L_-\) successively. To illustrate the process we’ll derive the 3 states for \(\ell = 1\). The top state \(|11\rangle\) can be obtained by following the derivation given in Shankar from his equation 12.5.28 onwards. In spherical coordinates, the raising and lowering operators have the form

\[
L_\pm = \pm \hbar e^{\pm i\phi} \left[ \frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \phi} \right] \tag{3}
\]

Applying \(L_+\) to the top state \(|11\rangle\) must give zero, so if \(\psi_1\) is the representation of this state in spherical coordinates, we must solve the differential equation

\[
\left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \psi_1 = 0 \tag{4}
\]

Since \(\psi_1\) is also an eigenfunction of \(L_z\) with eigenvalue \(\hbar\), we know that

\[
\psi_1 = U_1^1(r, \theta) e^{i\phi} \tag{5}
\]

Thus\(4\) becomes

\[
\left( \frac{\partial}{\partial \theta} - \cot \theta \right) U_1^1 = 0 \tag{6}
\]
This can be solved by writing it in the form

\[
\frac{dU_1^1}{U_1^1} = \frac{d(\sin \theta)}{\sin \theta} \quad (7)
\]

\[
\ln U_1^1 = \ln (\sin \theta) + \ln R(r) + \ln A \quad (8)
\]

where \( R \) is some unspecified function of \( r \), and \( A \) is a constant. We therefore have

\[
U_1^1(r, \theta) = R(r)(A \sin \theta) \quad (9)
\]

If we ignore \( R \) for now, we can normalize over the angular coordinates by requiring

\[
\int |A \sin \theta|^2 d\Omega = 1 \quad (10)
\]

The element \( d\Omega \) of solid angle is

\[
d\Omega = \sin \theta d\phi d\theta \quad (11)
\]

so we have

\[
|A|^2 \int_0^\pi \int_0^{2\pi} \sin^3 \theta d\phi d\theta = 2\pi |A|^2 \int_0^\pi \sin \theta (1 - \cos^2 \theta) d\theta \quad (12)
\]

\[
= \frac{8\pi}{3} |A|^2 \quad (13)
\]

\[
A = \sqrt{\frac{3}{8\pi}} \quad (14)
\]

Thus the spherical harmonic \( Y_1^1 \) is (using Shankar’s normalization convention of multiplying by \((-1)\ell\)):

\[
Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \quad (15)
\]

We can now get \( Y_1^0 \) by applying \( L_- \) to \( Y_1^1 \). From \ref{2} we have

\[
L_- Y_1^1 = \hbar \sqrt{(1+1)(1-1+1)} Y_1^0 \quad (16)
\]

\[
= \sqrt{2} \hbar Y_1^0 \quad (17)
\]

From \ref{3} we have
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\[ L \cdot Y_1^1 = -\hbar e^{-i\phi} \left[ \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^1 \]

\[ = -\hbar e^{-i\phi} \left( -\sqrt{\frac{3}{8\pi}} \right) \left[ \cos \theta - i \cot \theta (i \sin \theta) \right] e^{i\phi} \]

\[ = 2\hbar \sqrt{\frac{3}{8\pi}} \cos \theta \]

(18) \hspace{1cm} (19) \hspace{1cm} (20)

Comparing the last two results gives

\[ \sqrt{2\hbar} Y_0^0 = 2\sqrt{\frac{3}{8\pi}} \cos \theta \]

\[ Y_0^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \]

(21) \hspace{1cm} (22)

Repeating the process, we get

\[ L \cdot Y_1^0 = \hbar \sqrt{(1 + 0)(1 - 0 + 1)} Y_1^{-1} \]

\[ = \sqrt{2\hbar} Y_1^{-1} \]

(23) \hspace{1cm} (24)

Also

\[ L \cdot Y_1^0 = -\hbar e^{-i\phi} \left[ \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^0 \]

\[ = -\hbar e^{-i\phi} \sqrt{\frac{3}{4\pi}} (- \sin \theta - 0) \]

\[ = \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi} \]

(25) \hspace{1cm} (26) \hspace{1cm} (27)

Thus

\[ \sqrt{2\hbar} Y_1^{-1} = \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi} \]

\[ Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi} \]

(28) \hspace{1cm} (29)

Comparing these results with Shankar’s equation 12.5.39 we see that they match. [This exercise is similar to one we did earlier, where we used the raising operator to generate spherical harmonics with higher values of \( m \).]
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