

SPHERICAL HARMONICS USING THE LOWERING OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The raising and lowering operators for angular momentum are

$$(1) \quad L_{\pm} \equiv L_x \pm iL_y$$

On a state $|\ell m\rangle$ in the basis of eigenstates of L^2 and L_z , they have the effect:

$$(2) \quad L_{\pm} |\ell m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle$$

This means that, if we can find the top state $|\ell \ell\rangle$, we can find the state for all lower values of m by applying L_- successively. To illustrate the process we'll derive the 3 states for $\ell = 1$. The top state $|11\rangle$ can be obtained by following the derivation given in Shankar from his equation 12.5.28 onwards. In spherical coordinates, the raising and lowering operators have the form

$$(3) \quad L_{\pm} = \pm \hbar e^{\pm i\phi} \left[\frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \phi} \right]$$

Applying L_+ to the top state $|11\rangle$ must give zero, so if ψ_1^1 is the representation of this state in spherical coordinates, we must solve the differential equation

$$(4) \quad \left[\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right] \psi_1^1 = 0$$

Since ψ_1^1 is also an eigenfunction of L_z with eigenvalue \hbar , we know that

$$(5) \quad \psi_1^1 = U_1^1(r, \theta) e^{i\phi}$$

Thus 4 becomes

$$(6) \quad \left(\frac{\partial}{\partial \theta} - \cot \theta \right) U_1^1 = 0$$

This can be solved by writing it in the form

$$(7) \quad \frac{dU_1^1}{U_1^1} = \frac{d(\sin \theta)}{\sin \theta}$$

$$(8) \quad \ln U_1^1 = \ln(\sin \theta) + \ln R(r) + \ln A$$

where R is some unspecified function of r , and A is a constant. We therefore have

$$(9) \quad U_1^1(r, \theta) = R(r) (A \sin \theta)$$

If we ignore R for now, we can normalize over the angular coordinates by requiring

$$(10) \quad \int |A \sin \theta|^2 d\Omega = 1$$

The element $d\Omega$ of solid angle is

$$(11) \quad d\Omega = \sin \theta d\phi d\theta$$

so we have

$$(12) \quad |A|^2 \int_0^\pi \int_0^{2\pi} \sin^3 \theta d\phi d\theta = 2\pi |A|^2 \int_0^\pi \sin \theta (1 - \cos^2 \theta) d\theta$$

$$(13) \quad = \frac{8\pi}{3} |A|^2$$

$$(14) \quad A = \sqrt{\frac{3}{8\pi}}$$

Thus the spherical harmonic Y_1^1 is (using Shankar's normalization convention of multiplying by $(-1)^\ell$):

$$(15) \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

We can now get Y_1^0 by applying L_- to Y_1^1 . From 2 we have

$$(16) \quad L_- Y_1^1 = \hbar \sqrt{(1+1)(1-1+1)} Y_1^0$$

$$(17) \quad = \sqrt{2} \hbar Y_1^0$$

From 3 we have

$$(18) \quad L_- Y_1^1 = -\hbar e^{-i\phi} \left[\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^1$$

$$(19) \quad = -\hbar e^{-i\phi} \left(-\sqrt{\frac{3}{8\pi}} \right) [\cos \theta - i \cot \theta (i \sin \theta)] e^{i\phi}$$

$$(20) \quad = 2\hbar \sqrt{\frac{3}{8\pi}} \cos \theta$$

Comparing the last two results gives

$$(21) \quad \sqrt{2} \hbar Y_1^0 = 2 \sqrt{\frac{3}{8\pi}} \cos \theta$$

$$(22) \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

Repeating the process, we get

$$(23) \quad L_- Y_1^0 = \hbar \sqrt{(1+0)(1-0+1)} Y_1^{-1}$$

$$(24) \quad = \sqrt{2} \hbar Y_1^{-1}$$

Also

$$(25) \quad L_- Y_1^0 = -\hbar e^{-i\phi} \left[\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right] Y_1^0$$

$$(26) \quad = -\hbar e^{-i\phi} \sqrt{\frac{3}{4\pi}} (-\sin \theta - 0)$$

$$(27) \quad = \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi}$$

Thus

$$(28) \quad \sqrt{2} \hbar Y_1^{-1} = \hbar \sqrt{\frac{3}{4\pi}} \sin \theta e^{-i\phi}$$

$$(29) \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}$$

Comparing these results with Shankar's equation 12.5.39 we see that they match. [This exercise is similar to one we did earlier, where we used the raising operator to generate spherical harmonics with higher values of m .]

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