

## ANGULAR MOMENTUM AND PARITY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.12.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The parity operator in 3-d reflects every point directly through the origin, so that a position vector  $\mathbf{r} \rightarrow -\mathbf{r}$ . In rectangular coordinates this means replacing each coordinate by its negative. In spherical coordinates, the angular coordinates change according to

$$(0.1) \quad \theta \rightarrow \pi - \theta$$

$$(0.2) \quad \phi \rightarrow \pi + \phi$$

If this isn't obvious, picture reflecting a vector  $\mathbf{r}$  through the origin. If the original vector makes an angle  $\theta$  with the  $z$  (vertical) axis, then the reflected vector makes an angle  $\theta$  with the  $-z$  axis, which is equivalent to an angle of  $\pi - \theta$  with the  $+z$  axis. The azimuthal angle  $\phi$  just gets rotated by  $\pi$  to lie on the other side of the  $z$  axis.

Using this, we can see that the parity operator  $\Pi$  commutes with both  $L^2$  and  $L_z$ , as follows. Since neither of these operators involves the radial coordinate, we can consider their effect on a function  $f(\theta, \phi)$ . Under parity, we have

$$(0.3) \quad \Pi f(\theta, \phi) \rightarrow f(\pi - \theta, \pi + \phi)$$

Thus the derivatives transform under parity according to

$$(0.4) \quad \frac{\partial f(\theta, \phi)}{\partial \theta} \rightarrow -\frac{\partial f(\pi - \theta, \pi + \phi)}{\partial \theta}$$

$$(0.5) \quad \frac{\partial f(\theta, \phi)}{\partial \phi} \rightarrow \frac{\partial f(\pi - \theta, \pi + \phi)}{\partial \phi}$$

The angular momentum operators are

$$(0.6) \quad L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$(0.7) \quad L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Thus the combined operation gives

$$(0.8)$$

$$L^2 \Pi f(\theta, \phi) \rightarrow -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] f(\pi - \theta, \pi + \phi)$$

$$(0.9)$$

$$= -\hbar^2 \left[ \frac{1}{\sin \theta} \left( -\frac{\partial}{\partial \theta} \right) \left( \sin \theta \left( -\frac{\partial}{\partial \theta} \right) \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] f(\pi - \theta, \pi + \phi)$$

$$(0.10)$$

$$= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] f(\pi - \theta, \pi + \phi)$$

$$(0.11)$$

$$= L^2 f(\pi - \theta, \pi + \phi)$$

If we apply  $\Pi$  to  $L^2$ , we have

$$(0.12)$$

$$\Pi [L^2 f(\theta, \phi)] = -\hbar^2 \left[ \frac{1}{\sin(\pi - \theta)} \left( -\frac{\partial}{\partial \theta} \right) \left( \sin(\pi - \theta) \left( -\frac{\partial}{\partial \theta} \right) \right) + \frac{1}{\sin^2(\pi - \theta)} \frac{\partial^2}{\partial \phi^2} \right] f(\pi - \theta, \pi + \phi)$$

$$(0.13)$$

$$= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] f(\pi - \theta, \pi + \phi)$$

$$(0.14)$$

$$= L^2 f(\pi - \theta, \pi + \phi)$$

Thus

$$(0.15) \quad [\Pi, L^2] = 0$$

where in the first line we used  $\sin(\pi - \theta) = \sin \theta$ .

Since  $L_z$  involves only a derivative with respect to  $\phi$  which doesn't change under parity, we have

$$(0.16) \quad [\Pi, L_z] = 0$$

Since  $\Pi$  commutes with both  $L^2$  and  $L_z$  it is possible to find a set of functions that are simultaneous eigenfunctions of all three operators. These functions turn out to be the same spherical harmonics that we've been using all along. We can show this by starting with the top spherical harmonic

$$(0.17) \quad Y_l^l = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l \theta$$

where we've included the  $(-1)^l$  to be consistent with Shankar's equation 12.5.32. Under parity, this transforms as

$$(0.18) \quad \Pi Y_l^l = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il(\pi+\phi)} \sin^l(\pi-\theta)$$

$$(0.19) \quad = (-1)^l e^{il\pi} \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l \theta$$

$$(0.20) \quad = (-1)^l Y_l^l$$

where we used  $e^{il\pi} = (-1)^l$  in the second line. Thus  $Y_l^l$  is an eigenfunction of  $\Pi$  with eigenvalue  $(-1)^l$ .

To show that the other spherical harmonics are also eigenfunctions, we can use the lowering operator  $L_-$ . In spherical coordinates, we have

$$(0.21) \quad L_- Y_l^m = \hbar \sqrt{(\ell+m)(\ell-m+1)} Y_l^{m-1}$$

The operator can be expressed as

$$(0.22) \quad L_- = -\hbar e^{-i\phi} \left[ \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right]$$

Under parity, we can transform 0.22 using  $\sin(\pi-\theta) = \sin \theta$  and  $\cos(\pi-\theta) = -\cos \theta$ , so that  $\cot(\pi-\theta) = -\cot \theta$ . We therefore have

$$(0.23) \quad \Pi L_- = -\hbar e^{-i(\pi+\phi)} \left[ -\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right]$$

$$(0.24) \quad = -\hbar e^{-i\phi} \left[ \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right]$$

$$(0.25) \quad = L_-$$

Thus  $L_-$  is unchanged by parity, which means that from 0.21,  $Y_l^{m-1}$  has the same parity as  $Y_l^m$ . Starting with  $Y_l^l$  and using the lowering operator successively to reduce the superscript index, we have therefore

$$(0.26) \quad \Pi Y_l^m = (-1)^l Y_l^m$$

Thus all spherical harmonics are also eigenfunctions of parity.