ANGULAR MOMENTUM AND PARITY

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.12.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The parity operator in 3-d reflects every point directly through the origin, so that a position vector $\mathbf{r}\to-\mathbf{r}$. In rectangular coordinates this means replacing each coordinate by its negative. In spherical coordinates, the angular coordinates change according to

$$\theta \to \pi-\theta \tag{1}$$
$$\phi \to \pi+\phi \tag{2}$$

If this isn't obvious, picture reflecting a vector $\mathbf{r}$ through the origin. If the original vector makes an angle θ with the z (vertical) axis, then the reflected vector makes an angle θ with the $-z$ axis, which is equivalent to an angle of $\pi-\theta$ with the $+z$ axis. The azimuthal angle ϕ just gets rotated by π to lie on the other side of the z axis.

Using this, we can see that the parity operator Π commutes with both L^2 and L_z, as follows. Since neither of these operators involves the radial coordinate, we can consider their effect on a function $f(\theta,\phi)$. Under parity, we have

$$\Pi f(\theta,\phi)\to f(\pi-\theta,\pi+\phi) \tag{3}$$

Thus the derivatives transform under parity according to

$$\frac{\partial f(\theta,\phi)}{\partial\theta} \to -\frac{\partial f(\pi-\theta,\pi+\phi)}{\partial\theta} \tag{4}$$
$$\frac{\partial f(\theta,\phi)}{\partial\phi} \to \frac{\partial f(\pi-\theta,\pi+\phi)}{\partial\phi} \tag{5}$$

The angular momentum operators are

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{6}$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{7}$$

Thus the combined operation gives

(8)

$$L^2\Pi f(\theta,\phi) \to -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f(\pi-\theta,\pi+\phi)$$

(9)

$$= -\hbar^2\left[\frac{1}{\sin\theta}\left(-\frac{\partial}{\partial\theta}\right)\left(\sin\theta\left(-\frac{\partial}{\partial\theta}\right)\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f(\pi-\theta,\pi+\phi)$$

(10)

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f(\pi-\theta,\pi+\phi)$$

(11)

$$= L^2 f(\pi-\theta,\pi+\phi)$$

If we apply Π to L^2, we have

(12)

$$\Pi\left[L^2 f(\theta,\phi)\right] = -\hbar^2\left[\frac{1}{\sin(\pi-\theta)}\left(-\frac{\partial}{\partial\theta}\right)\left(\sin(\pi-\theta)\left(-\frac{\partial}{\partial\theta}\right)\right)+\frac{1}{\sin^2(\pi-\theta)}\frac{\partial^2}{\partial\phi^2}\right]f(\pi-\theta,\pi$$

(13)

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f(\pi-\theta,\pi+\phi)$$

(14)

$$= L^2 f(\pi-\theta,\pi+\phi)$$

Thus

$$\left[\Pi, L^2\right] = 0 \tag{15}$$

where in the first line we used $\sin(\pi-\theta) = \sin\theta$.

Since L_z involves only a derivative with respect to ϕ which doesn't change under parity, we have

$$[\Pi, L_z] = 0 \tag{16}$$

Since Π commutes with both L^2 and L_z it is possible to find a set of functions that are simultaneous eigenfunctions of all three operators. These functions turn out to be the same spherical harmonics that we've been using all along. We can show this by starting with the top spherical harmonic

$$Y_l^l = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l \theta \tag{17}$$

where we've included the $(-1)^l$ to be consistent with Shankar's equation 12.5.32. Under parity, this transforms as

$$\Pi Y_l^l = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il(\pi+\phi)} \sin^l (\pi-\theta) \tag{18}$$

$$= (-1)^l e^{il\pi} \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l \theta \tag{19}$$

$$= (-1)^l Y_l^l \tag{20}$$

where we used $e^{il\pi} = (-1)^l$ in the second line. Thus Y_l^l is an eigenfunction of Π with eigenvalue $(-1)^l$.

To show that the other spherical harmonics are also eigenfunctions, we can use the lowering operator L_-. In spherical coordinates, we have

$$L_- Y_l^m = \hbar \sqrt{(\ell+m)(\ell-m+1)} Y_l^{m-1} \tag{21}$$

The operator can be expressed as

$$L_- = -\hbar e^{-i\phi} \left[\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\phi} \right] \tag{22}$$

Under parity, we can transform 22 using $\sin(\pi-\theta) = \sin\theta$ and $\cos(\pi-\theta) = -\cos\theta$, so that $\cot(\pi-\theta) = -\cot\theta$. We therefore have

$$\Pi L_- = -\hbar e^{-i(\pi+\phi)} \left[-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right] \tag{23}$$

$$= -\hbar e^{-i\phi} \left[\frac{\partial}{\partial\theta} - i \cot\theta \frac{\partial}{\partial\phi} \right] \tag{24}$$

$$= L_- \tag{25}$$

Thus L_- is unchanged by parity, which means that from 21, Y_l^{m-1} has the same parity as Y_l^m. Starting with Y_l^l and using the lowering operator successively to reduce the superscript index, we have therefore

$$\Pi Y_l^m = (-1)^l Y_l^m \tag{26}$$

Thus all spherical harmonics are also eigenfunctions of parity.