

LINEAR COMBINATIONS OF SPHERICAL HARMONICS - PROBABILITIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.
Chapter 12, Exercise 12.5.13.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

If we can express a 3-d quantum state in terms of the spherical harmonics, we can calculate directly the probabilities of L_z having one of its eigenvalues. That is, if we can write a state ψ as

$$(0.1) \quad \psi(r, \theta, \phi) = f(r) \sum_m C_l^m Y_l^m$$

for some constant coefficients C_l^m and f is some function of r alone, then

$$(0.2) \quad P(l_z = m\hbar) = \frac{|C_l^m|^2}{\sum_n |C_l^n|^2}$$

As an example, suppose we have

$$(0.3) \quad \psi = N(x + y + 2z) e^{-\alpha r}$$

where N is a normalization constant. We start by expressing x , y and z in terms of Y_l^m . We have

$$(0.4) \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$(0.5) \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

Using standard spherical-to-rectangular conversions

$$(0.6) \quad x = r \sin \theta \cos \phi$$

$$(0.7) \quad y = r \sin \theta \sin \phi$$

$$(0.8) \quad z = r \cos \theta$$

Therefore

$$(0.9) \quad \cos \phi = \frac{x}{r \sin \theta}$$

$$(0.10) \quad \sin \phi = \frac{y}{r \sin \theta}$$

$$(0.11) \quad \cos \theta = \frac{z}{r}$$

Plugging these into 0.4 we have

$$(0.12) \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta (\cos \phi \pm i \sin \phi)$$

$$(0.13) \quad = \mp \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$$

$$(0.14) \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$(0.15) \quad = \sqrt{2} \sqrt{\frac{3}{8\pi}} \frac{z}{r}$$

Inverting these, we have

$$(0.16) \quad x = \frac{1}{2} \sqrt{\frac{8\pi}{3}} r (Y_1^{-1} - Y_1^1)$$

$$(0.17) \quad y = -\frac{1}{2i} \sqrt{\frac{8\pi}{3}} r (Y_1^{-1} + Y_1^1)$$

$$(0.18) \quad z = \frac{1}{\sqrt{2}} \sqrt{\frac{8\pi}{3}} r Y_1^0$$

Thus 0.3 becomes

$$(0.19) \quad \psi = \sqrt{\frac{8\pi}{3}} N r e^{-\alpha r} \left[Y_1^1 \left(-\frac{1}{2} - \frac{1}{2i} \right) + Y_1^{-1} \left(\frac{1}{2} - \frac{1}{2i} \right) + Y_1^0 \sqrt{2} \right]$$

Comparing with 0.1 we find

$$(0.20) \quad C_1^1 = -\frac{1}{2} - \frac{1}{2i}$$

$$(0.21) \quad C_1^{-1} = \frac{1}{2} - \frac{1}{2i}$$

$$(0.22) \quad C_1^0 = \sqrt{2}$$

We have

$$(0.23) \quad \sum_n |C_l^n|^2 = \frac{1}{2} + \frac{1}{2} + 2 = 3$$

$$(0.24) \quad P(l_z = 0) = \frac{|C_l^0|^2}{\sum_n |C_l^n|^2} = \frac{2}{3}$$

$$(0.25) \quad P(l_z = \hbar) = \frac{|C_l^1|^2}{\sum_n |C_l^n|^2} = \frac{1}{6}$$

$$(0.26) \quad P(l_z = -\hbar) = \frac{|C_l^{-1}|^2}{\sum_n |C_l^n|^2} = \frac{1}{6}$$

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