# LINEAR COMBINATIONS OF SPHERICAL HARMONICS - PROBABILITIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.13.

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If we can express a 3-d quantum state in terms of the spherical harmonics, we can calculate directly the probabilities of $L_z$ having one of its eigenvalues. That is, if we can write a state $\psi$ as

$$\psi(r,\theta,\phi)=f(r)\sum_m C_l^m Y_l^m \tag{1}$$

for some constant coefficients $C_l^m$ and $f$ is some function of $r$ alone, then

$$P(l_z=m\hbar)=\frac{\left|C_l^m\right|^2}{\sum_n\left|C_l^n\right|^2} \tag{2}$$

As an example, suppose we have

$$\psi=N(x+y+2z)e^{-\alpha r} \tag{3}$$

where $N$ is a normalization constant. We start by expressing $x$, $y$ and $z$ in terms of $Y_1^m$. We have

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{4}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{5}$$

Using standard spherical-to-rectangular conversions

$$x = r\sin\theta\cos\phi \tag{6}$$

$$y = r\sin\theta\sin\phi \tag{7}$$

$$z = r\cos\theta \tag{8}$$

Therefore

$$\cos\phi = \frac{x}{r\sin\theta} \tag{9}$$
$$\sin\phi = \frac{y}{r\sin\theta} \tag{10}$$
$$\cos\theta = \frac{z}{r} \tag{11}$$

Plugging these into 4 we have

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\left(\cos\phi \pm i\sin\phi\right) \tag{12}$$
$$= \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r} \tag{13}$$
$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \tag{14}$$
$$= \sqrt{2}\sqrt{\frac{3}{8\pi}}\frac{z}{r} \tag{15}$$

Inverting these, we have

$$x = \frac{1}{2}\sqrt{\frac{8\pi}{3}}r\left(Y_1^{-1} - Y_1^1\right) \tag{16}$$
$$y = -\frac{1}{2i}\sqrt{\frac{8\pi}{3}}r\left(Y_1^{-1} + Y_1^1\right) \tag{17}$$
$$z = \frac{1}{\sqrt{2}}\sqrt{\frac{8\pi}{3}}rY_1^0 \tag{18}$$

Thus 3 becomes

$$\psi = \sqrt{\frac{8\pi}{3}}Nre^{-\alpha r}\left[Y_1^1\left(-\frac{1}{2}-\frac{1}{2i}\right) + Y_1^{-1}\left(\frac{1}{2}-\frac{1}{2i}\right) + Y_1^0\sqrt{2}\right] \tag{19}$$

Comparing with 1 we find

$$C_1^1 = -\frac{1}{2}-\frac{1}{2i} \tag{20}$$
$$C_1^{-1} = \frac{1}{2}-\frac{1}{2i} \tag{21}$$
$$C_1^0 = \sqrt{2} \tag{22}$$

We have

$$\sum_n |C_1^n|^2 = \frac{1}{2}+\frac{1}{2}+2=3 \tag{23}$$

$$P(l_z=0)=\frac{\left|C_l^0\right|^2}{\sum_n\left|C_l^n\right|^2}=\frac{2}{3} \tag{24}$$

$$P(l_z=\hbar)=\frac{\left|C_l^1\right|^2}{\sum_n\left|C_l^n\right|^2}=\frac{1}{6} \tag{25}$$

$$P(l_z=-\hbar)=\frac{\left|C_l^{-1}\right|^2}{\sum_n\left|C_l^n\right|^2}=\frac{1}{6} \tag{26}$$

## PINGBACKS

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