

LINEAR COMBINATIONS OF SPHERICAL HARMONICS - PROBABILITIES

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.
Chapter 12, Exercise 12.5.13.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

If we can express a 3-d quantum state in terms of the spherical harmonics, we can calculate directly the probabilities of L_z having one of its eigenvalues. That is, if we can write a state ψ as

$$\psi(r, \theta, \phi) = f(r) \sum_m C_l^m Y_l^m \quad (1)$$

for some constant coefficients C_l^m and f is some function of r alone, then

$$P(l_z = m\hbar) = \frac{|C_l^m|^2}{\sum_n |C_l^n|^2} \quad (2)$$

As an example, suppose we have

$$\psi = N(x + y + 2z)e^{-\alpha r} \quad (3)$$

where N is a normalization constant. We start by expressing x , y and z in terms of Y_l^m . We have

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \quad (4)$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \quad (5)$$

Using standard spherical-to-rectangular conversions

$$x = r \sin\theta \cos\phi \quad (6)$$

$$y = r \sin\theta \sin\phi \quad (7)$$

$$z = r \cos\theta \quad (8)$$

Therefore

$$\cos \phi = \frac{x}{r \sin \theta} \quad (9)$$

$$\sin \phi = \frac{y}{r \sin \theta} \quad (10)$$

$$\cos \theta = \frac{z}{r} \quad (11)$$

Plugging these into 4 we have

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta (\cos \phi \pm i \sin \phi) \quad (12)$$

$$= \mp \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r} \quad (13)$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \quad (14)$$

$$= \sqrt{2} \sqrt{\frac{3}{8\pi}} \frac{z}{r} \quad (15)$$

Inverting these, we have

$$x = \frac{1}{2} \sqrt{\frac{8\pi}{3}} r (Y_1^{-1} - Y_1^1) \quad (16)$$

$$y = -\frac{1}{2i} \sqrt{\frac{8\pi}{3}} r (Y_1^{-1} + Y_1^1) \quad (17)$$

$$z = \frac{1}{\sqrt{2}} \sqrt{\frac{8\pi}{3}} r Y_1^0 \quad (18)$$

Thus 3 becomes

$$\psi = \sqrt{\frac{8\pi}{3}} N r e^{-\alpha r} \left[Y_1^1 \left(-\frac{1}{2} - \frac{1}{2i} \right) + Y_1^{-1} \left(\frac{1}{2} - \frac{1}{2i} \right) + Y_1^0 \sqrt{2} \right] \quad (19)$$

Comparing with 1 we find

$$C_1^1 = -\frac{1}{2} - \frac{1}{2i} \quad (20)$$

$$C_1^{-1} = \frac{1}{2} - \frac{1}{2i} \quad (21)$$

$$C_1^0 = \sqrt{2} \quad (22)$$

We have

$$\sum_n |C_l^n|^2 = \frac{1}{2} + \frac{1}{2} + 2 = 3 \quad (23)$$

$$P(l_z = 0) = \frac{|C_l^0|^2}{\sum_n |C_l^n|^2} = \frac{2}{3} \quad (24)$$

$$P(l_z = \hbar) = \frac{|C_l^1|^2}{\sum_n |C_l^n|^2} = \frac{1}{6} \quad (25)$$

$$P(l_z = -\hbar) = \frac{|C_l^{-1}|^2}{\sum_n |C_l^n|^2} = \frac{1}{6} \quad (26)$$

PINGBACKS

Pingback: spherical harmonics: rotation about the x axis

Pingback: Wigner-Eckart Theorem