SPHERICALLY SYMMETRIC POTENTIALS: A SIMPLE EXAMPLE

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.1.

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The Schrödinger equation in 3-d for a potential that depends only on r is

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]+V\psi=E\psi \tag{1}$$

The angular part of the operator on the LHS is essentially the angular momentum operator L^2 (times $1/2\mu r^2$):

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{2}$$

, so we can write this as

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+V\psi+\frac{L^2}{2\mu r^2}\psi=E\psi \tag{3}$$

Eigenfunctions in this equation satisfy

$$\psi=R_{Elm}(r)Y_l^m(\theta,\phi) \tag{4}$$

where the subscript Elm refers to the energy E and the angular momentum quantum numbers l and m. Y_l^m is a spherical harmonic and R_{Elm} is the radial function which depends on the potential V. The eigenvalues of L^2 are $l(l+1)\hbar^2$ so 3 becomes

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{El}}{\partial r}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}R_{El}+VR_{El}=ER_{El} \tag{5}$$

We've dropped the m from R_{Elm} since, for a spherically symmetric potential, the radial function is independent of m.

Example. Suppose a particle is described by the wave function

$$\psi_E(r,\theta,\phi) = Ae^{-r/a_0} \tag{6}$$

where A and a_0 are constants. What can we deduce about the system? First, since ψ_E is independent of θ and ϕ we see from 2 that

$$L^2\psi_E = 0 \tag{7}$$

so the eigenvalue is $l = 0$ and the state has no angular momentum. From 3 we therefore have

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + V\psi = E\psi \tag{8}$$

Working out the derivatives, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = -\frac{A}{r^2}\frac{d}{dr}\left(\frac{r^2}{a_0}e^{-r/a_0}\right) \tag{9}$$

$$= Ae^{-r/a_0}\left(-\frac{2}{ra_0} + \frac{1}{a_0^2}\right) \tag{10}$$

Plugging this back into 8 and cancelling terms gives

$$-\frac{2}{ra_0} + \frac{1}{a_0^2} = \frac{2\mu}{\hbar^2}(V - E) \tag{11}$$

If $V(r) \to 0$ as $r \to \infty$ we have, in this limit

$$E = -\frac{\hbar^2}{2\mu a_0^2} \tag{12}$$

The energy is constant at all values of r so we can now find V from 11

$$-\frac{2}{ra_0} + \frac{1}{a_0^2} = \frac{2\mu}{\hbar^2}\left(V(r) + \frac{\hbar^2}{2\mu a_0^2}\right) \tag{13}$$

$$V(r) = -\frac{\hbar^2}{\mu a_0 r} \tag{14}$$