

NONDEGENERATE STATES IN 3-D: SPHERICALLY SYMMETRIC SYSTEMS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.5.

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In solving the Schrödinger equation for spherically symmetric potentials, we found that we could reduce the problem to the equation

$$(0.1) \quad \left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \right] U_{El} = EU_{El}$$

where $U_{El}(r)$ is related to the radial function by

$$(0.2) \quad R_{El}(r) = \frac{U_{El}(r)}{r}$$

We can write 0.1 as an eigenvalue equation for the operator D_l in the form

$$(0.3) \quad D_l(r) U_{El} = EU_{El}$$

with

$$(0.4) \quad D_l(r) \equiv -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}$$

We can show that, provided $U_{El}(r) \rightarrow 0$ as $r \rightarrow 0$, there are no degenerate eigenstates (that is, any state U_{El} that is an eigenstate with energy E is unique up to a scaling factor). The proof is similar to that in 1-d quantum mechanics, and goes by contradiction.

We suppose that there are two different functions U_1 and U_2 that satisfy 0.1 for the same energy E (and same angular momentum number l). We then have

$$(0.5) \quad \left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \right] U_1 = EU_1$$

$$(0.6) \quad \left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \right] U_2 = EU_2$$

Multiply the first by U_2 and the second by U_1 and subtract to get

$$(0.7) \quad U_2 U_1'' - U_1 U_2'' = 0$$

This expression is

$$(0.8) \quad U_2 U_1'' - U_1 U_2'' = \frac{d}{dr} (U_2 U_1' - U_1 U_2') = 0$$

which we can integrate to get

$$(0.9) \quad U_2 U_1' - U_1 U_2' = C$$

for some constant C . This relation is valid for all r , so we can choose $r = 0$ where $U_2(0) = U_1(0) = 0$, which shows that $C = 0$. Therefore

$$(0.10) \quad \frac{U_1'}{U_1} = \frac{U_2'}{U_2}$$

Integrating gives us

$$(0.11) \quad \ln U_1 = \ln U_2 + K$$

for some other constant K , so

$$(0.12) \quad U_1 = e^K U_2$$

That is, any two eigenfunctions with the same eigenvalue E are multiples of each other, so represent the same state, which is nondegenerate.

Note that the derivation didn't rely on the value of U anywhere except at $r = 0$, so there is no requirement that, for example, $U \rightarrow 0$ as $r \rightarrow \infty$. Also, the derivation is valid whatever the sign of E .