NONDEGENERATE STATES IN 3-D: SPHERICALLY SYMMETRIC SYSTEMS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.5.

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In solving the Schrödinger equation for spherically symmetric potentials, we found that we could reduce the problem to the equation

$$\left[-\frac{\hbar^{2}}{2\mu}\frac{d^{2}}{dr^{2}}+V\left(r\right)+\frac{l\left(l+1\right)\hbar^{2}}{2\mu r^{2}}\right]U_{El}=EU_{El} \tag{1}$$

where $U_{El}\left(r\right)$ is related to the radial function by

$$R_{El}\left(r\right)=\frac{U_{El}\left(r\right)}{r} \tag{2}$$

We can write 1 as an eigenvalue equation for the operator D_{l} in the form

$$D_{l}\left(r\right)U_{El}=EU_{El} \tag{3}$$

with

$$D_{l}\left(r\right)\equiv-\frac{\hbar^{2}}{2\mu}\frac{d^{2}}{dr^{2}}+V\left(r\right)+\frac{l\left(l+1\right)\hbar^{2}}{2\mu r^{2}} \tag{4}$$

We can show that, provided $U_{El}\left(r\right)\to0$ as $r\to0$, there are no degenerate eigenstates (that is, any state U_{El} that is an eigenstate with energy E is unique up to a scaling factor). The proof is similar to that in 1-d quantum mechanics, and goes by contradiction.

We suppose that there are two different functions U_{1} and U_{2} that satisfy 1 for the same energy E (and same angular momentum number l). We then have

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V(r)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right]U_1=EU_1 \tag{5}$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V(r)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right]U_2=EU_2 \tag{6}$$

Multiply the first by U_2 and the second by U_1 and subtract to get

$$U_2U_1''-U_1U_2''=0 \tag{7}$$

This expression is

$$U_2U_1''-U_1U_2''=\frac{d}{dr}\left(U_2U_1'-U_1U_2'\right)=0 \tag{8}$$

which we can integrate to get

$$U_2U_1'-U_1U_2'=C \tag{9}$$

for some constant C. This relation is valid for all r, so we can choose $r=0$ where $U_2(0)=U_1(0)=0$, which shows that $C=0$. Therefore

$$\frac{U_1'}{U_1}=\frac{U_2'}{U_2} \tag{10}$$

Integrating gives us

$$\ln U_1=\ln U_2+K \tag{11}$$

for some other constant K, so

$$U_1=e^KU_2 \tag{12}$$

That is, any two eigenfunctions with the same eigenvalue E are multiples of each other, so represent the same state, which is nondegenerate.

Note that the derivation didn't rely on the value of U anywhere except at $r=0$, so there is no requirement that, for example, $U\to 0$ as $r\to\infty$. Also, the derivation is valid whatever the sign of E.