SPHERICAL BESSEL FUNCTIONS: BEHAVIOUR FOR SMALL ARGUMENTS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.7.

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The general solution for a free particle in spherical coordinates involves the radial function, which turns out to be

$$\frac{R_{l+1}}{\rho^{l+1}}=\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^{l+1}\frac{R_0}{\rho^0} \tag{1}$$

where l is the total angular momentum quantum number and

$$k^2 \equiv \frac{2\mu E}{\hbar^2} \tag{2}$$

$$\rho \equiv kr \tag{3}$$

We can rewrite this as

$$R_l=(-\rho)^l\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l R_0 \tag{4}$$

We saw earlier that the solutions for $l=0$ are, with $U_l=\rho R_l$

$$U_0^A(\rho) = \sin\rho \tag{5}$$

$$U_0^B(\rho) = -\cos\rho \tag{6}$$

Thus the two solutions for $l=0$ are

$$R_0^A = \frac{\sin\rho}{\rho} \tag{7}$$

$$R_0^B = -\frac{\cos\rho}{\rho} \tag{8}$$

From these starting points, we can generate all the solutions for higher values of l using 4. These functions are

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho} \tag{9}$$

$$n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\cos\rho}{\rho} \tag{10}$$

and are known as spherical Bessel functions j_l and spherical Neumann functions n_l.

The asymptotic behaviour is given by

$$j_l \underset{\rho\to\infty}{\longrightarrow} \frac{1}{\rho}\sin\left(\rho - \frac{l\pi}{2}\right) \tag{11}$$

$$n_l \underset{\rho\to\infty}{\longrightarrow} -\frac{1}{\rho}\cos\left(\rho - \frac{l\pi}{2}\right) \tag{12}$$

For $\rho \to 0$, we have

$$j_l \underset{\rho\to 0}{\longrightarrow} \frac{\rho^l}{(2l+1)!!} \tag{13}$$

$$n_l \underset{\rho\to 0}{\longrightarrow} -\frac{(2l-1)!!}{\rho^{l+1}} \tag{14}$$

We can verify the latter equation for j_l for a couple of cases with small l. From 9, we can generate a couple of j_ls:

$$j_0 = \frac{\sin\rho}{\rho} \tag{15}$$

$$j_1 = -\rho\frac{1}{\rho}\frac{d}{dr}\left(\frac{\sin\rho}{\rho}\right) \tag{16}$$

$$= -\rho\frac{1}{\rho}\left(\frac{\cos\rho}{\rho} - \frac{\sin\rho}{\rho^2}\right) \tag{17}$$

$$= \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \tag{18}$$

$$j_2 = (-\rho)^2\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{dr}\left(\frac{\sin\rho}{\rho}\right)\right] \tag{19}$$

$$(-\rho)^2\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{1}{\rho}\left(\frac{\cos\rho}{\rho} - \frac{\sin\rho}{\rho^2}\right)\right] \tag{20}$$

$$= \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \frac{3\cos\rho}{\rho^2} \tag{21}$$

We can get the limits for $\rho \to 0$ by expanding the sine and cosine. That is, we use the limiting forms

$$\sin\rho \quad \to \quad \rho - \frac{\rho^3}{3!} + \ldots \tag{22}$$

$$\cos\rho \quad \to \quad 1 - \frac{1}{2}\rho^2 + \ldots \tag{23}$$

We need to retain enough terms for j_l so that we get all the terms up to the first power of ρ that doesn't cancel out when we do the algebra. We get

$$j_0 \to 1 = \frac{\rho^0}{1!!} \tag{24}$$

$$j_1 \to \frac{1}{\rho} - \frac{\rho}{6} - \frac{1}{\rho}\left(1 - \frac{1}{2}\rho^2\right) \tag{25}$$

$$= \frac{\rho}{3} = \frac{\rho^1}{3!!} \tag{26}$$

$$j_2 \to \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\left(\rho - \frac{\rho^3}{6} + \frac{\rho^5}{120}\right) - \frac{3}{\rho^2}\left(1 - \frac{1}{2}\rho^2 + \frac{1}{24}\rho^4\right) \tag{27}$$

$$\to \left(\frac{1}{6} + \frac{1}{40} - \frac{1}{8}\right)\rho^2 \tag{28}$$

$$= \frac{20 + 3 - 15}{120}\rho^2 \tag{29}$$

$$= \frac{\rho^2}{15} = \frac{\rho^2}{5!!} \tag{30}$$

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