

SPHERICAL BESSEL FUNCTIONS: BEHAVIOUR FOR SMALL ARGUMENTS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.7.

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The general solution for a free particle in spherical coordinates involves the radial function, which turns out to be

$$\frac{R_{l+1}}{\rho^{l+1}} = \left(-\frac{1}{\rho} \frac{d}{d\rho} \right)^{l+1} \frac{R_0}{\rho^0} \quad (1)$$

where l is the total angular momentum quantum number and

$$k^2 \equiv \frac{2\mu E}{\hbar^2} \quad (2)$$

$$\rho \equiv kr \quad (3)$$

We can rewrite this as

$$R_l = (-\rho)^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l R_0 \quad (4)$$

We saw earlier that the solutions for $l = 0$ are, with $U_l = \rho R_l$

$$U_0^A(\rho) = \sin \rho \quad (5)$$

$$U_0^B(\rho) = -\cos \rho \quad (6)$$

Thus the two solutions for $l = 0$ are

$$R_0^A = \frac{\sin \rho}{\rho} \quad (7)$$

$$R_0^B = -\frac{\cos \rho}{\rho} \quad (8)$$

From these starting points, we can generate all the solutions for higher values of l using 4. These functions are

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l \frac{\sin \rho}{\rho} \quad (9)$$

$$n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l \frac{\cos \rho}{\rho} \quad (10)$$

and are known as spherical Bessel functions j_l and spherical Neumann functions n_l .

The asymptotic behaviour is given by

$$j_l \xrightarrow{\rho \rightarrow \infty} \frac{1}{\rho} \sin \left(\rho - \frac{l\pi}{2} \right) \quad (11)$$

$$n_l \xrightarrow{\rho \rightarrow \infty} -\frac{1}{\rho} \cos \left(\rho - \frac{l\pi}{2} \right) \quad (12)$$

For $\rho \rightarrow 0$, we have

$$j_l \xrightarrow{\rho \rightarrow 0} \frac{\rho^l}{(2l+1)!!} \quad (13)$$

$$n_l \xrightarrow{\rho \rightarrow 0} -\frac{(2l-1)!!}{\rho^{l+1}} \quad (14)$$

We can verify the latter equation for j_l for a couple of cases with small l . From 9, we can generate a couple of j_l s:

$$j_0 = \frac{\sin \rho}{\rho} \quad (15)$$

$$j_1 = -\rho \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{\sin \rho}{\rho} \right) \quad (16)$$

$$= -\rho \frac{1}{\rho} \left(\frac{\cos \rho}{\rho} - \frac{\sin \rho}{\rho^2} \right) \quad (17)$$

$$= \frac{\sin \rho}{\rho^2} - \frac{\cos \rho}{\rho} \quad (18)$$

$$j_2 = (-\rho)^2 \frac{1}{\rho} \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\frac{\sin \rho}{\rho} \right) \right] \quad (19)$$

$$(-\rho)^2 \frac{1}{\rho} \frac{d}{d\rho} \left[\frac{1}{\rho} \left(\frac{\cos \rho}{\rho} - \frac{\sin \rho}{\rho^2} \right) \right] \quad (20)$$

$$= \left(\frac{3}{\rho^3} - \frac{1}{\rho} \right) \sin \rho - \frac{3 \cos \rho}{\rho^2} \quad (21)$$

We can get the limits for $\rho \rightarrow 0$ by expanding the sine and cosine. That is, we use the limiting forms

$$\sin \rho \rightarrow \rho - \frac{\rho^3}{3!} + \dots \quad (22)$$

$$\cos \rho \rightarrow 1 - \frac{1}{2}\rho^2 + \dots \quad (23)$$

We need to retain enough terms for j_l so that we get all the terms up to the first power of ρ that doesn't cancel out when we do the algebra. We get

$$j_0 \rightarrow 1 = \frac{\rho^0}{1!!} \quad (24)$$

$$j_1 \rightarrow \frac{1}{\rho} - \frac{\rho}{6} - \frac{1}{\rho} \left(1 - \frac{1}{2}\rho^2\right) \quad (25)$$

$$= \frac{\rho}{3} = \frac{\rho^1}{3!!} \quad (26)$$

$$j_2 \rightarrow \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right) \left(\rho - \frac{\rho^3}{6} + \frac{\rho^5}{120}\right) - \frac{3}{\rho^2} \left(1 - \frac{1}{2}\rho^2 + \frac{1}{24}\rho^4\right) \quad (27)$$

$$\rightarrow \left(\frac{1}{6} + \frac{1}{40} - \frac{1}{8}\right) \rho^2 \quad (28)$$

$$= \frac{20 + 3 - 15}{120} \rho^2 \quad (29)$$

$$= \frac{\rho^2}{15} = \frac{\rho^2}{5!!} \quad (30)$$

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