

## ISOTROPIC HARMONIC OSCILLATOR IN 3-D: USE OF SPHERICAL HARMONICS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've solved the 3-d isotropic harmonic oscillator before, so we've already solved most of Shankar's exercise 12.6.11. We can quote the results here. The solution has the form

$$(0.1) \quad \psi_{Elm} = \frac{U_{El}(r)}{r} Y_l^m(\theta, \phi)$$

The earlier solution uses notation from Griffiths's book, but as the end result is the same, it's not worth going through the derivation again using Shankar's notation.

The potential is

$$(0.2) \quad V(r) = \frac{1}{2} m \omega^2 r^2$$

The radial equation to be solved is

$$(0.3) \quad \frac{d^2 u}{d\rho^2} = \left( -1 + \frac{l(l+1)}{\rho^2} + \rho_0^2 \rho^2 \right) u$$

If we define

$$(0.4) \quad \kappa^2 \equiv \frac{2\mu E}{\hbar^2}$$

$$(0.5) \quad \rho \equiv \kappa r$$

$$(0.6) \quad \rho_0 \equiv \frac{\mu \omega}{\hbar \kappa^2} = \frac{\hbar \omega}{2E}$$

Taking the asymptotic behaviour of the radial function for small and large  $r$  into account leads us to a solution of form

$$(0.7) \quad u(\rho) = \rho^{l+1} e^{-\rho_0 \rho^2 / 2} v(\rho)$$

Note that Griffiths's  $v$  is not the same as Shankar's  $v$ , the latter of which is defined by Shankar's equation 12.6.49.

This gives a differential equation for Griffiths's  $v$

$$(0.8) \quad \rho \frac{d^2 v}{d\rho^2} + 2(l+1 - \rho_0 \rho^2) \frac{dv}{d\rho} + \rho(1 - \rho_0(2l+3))v = 0$$

The function  $v$  can be solved as a power series, giving

$$(0.9) \quad v(\rho) = \sum c_j \rho^j$$

Substituting into 0.8 leads to the recursion relation

$$(0.10) \quad c_{q+2} = \frac{\rho_0(2q+2l+3) - 1}{(q+2)(q+2l+3)} c_q$$

with  $c_1 = 0$ , so that  $c_q = 0$  for all odd  $q$ . The requirement that the series terminates at some finite value of  $j$  leads to the quantization condition on  $E$ :

$$(0.11) \quad E = \hbar\omega \left( q_{max} + l + \frac{3}{2} \right)$$

or, defining  $n = q_{max} + l$ ,

$$(0.12) \quad E_n = \hbar\omega \left( n + \frac{3}{2} \right)$$

We worked out the degeneracies in the earlier post as well, so that the degeneracy of  $E_n$  is

$$(0.13) \quad d(n) = \frac{1}{2}(n+1)(n+2)$$

To complete Shankar's exercise, we need to work out the eigenfunctions for  $n = 0$  and  $n = 1$ . For  $n = 0$ ,  $q_{max} = l = 0$ , so only  $c_0 \neq 0$  and we have

$$(0.14) \quad v(\rho) = c_0$$

$$(0.15) \quad u(\rho) = c_0 \rho e^{-\rho_0 \rho^2/2}$$

$$(0.16) \quad \psi_{000} = \frac{u}{r} Y_0^0$$

$$(0.17) \quad = c_0 \kappa e^{-\rho_0 \rho^2/2} Y_0^0$$

$$(0.18) \quad = c_0 \sqrt{\frac{2\mu 3\omega}{4\pi\hbar}} e^{-\mu\omega r^2/2\hbar}$$

where in the fourth line we used

$$(0.19) \quad \kappa = \frac{\sqrt{2\mu E}}{\hbar} = \frac{\sqrt{2\mu \frac{3}{2}\hbar\omega}}{\hbar} = \sqrt{\frac{3\mu\omega}{\hbar}}$$

$$(0.20) \quad Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

Normalizing this requires that

$$(0.21) \quad \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_{000}^2 r^2 \sin\theta dr d\theta d\phi = c_0^2 \frac{6\mu\omega}{\hbar} \int_0^\infty e^{-\mu\omega r^2/\hbar} r^2 dr$$

$$(0.22) \quad = 1$$

This is a standard Gaussian integral and can be done using software or tables so we get

$$(0.23) \quad c_0 = \frac{\sqrt{6}}{3} \left( \frac{\mu\omega}{\pi\hbar} \right)^{1/4}$$

This gives a wave function of

$$(0.24) \quad \psi_{000} = \left( \frac{\mu\omega}{\pi\hbar} \right)^{3/4} e^{-\mu\omega r^2/2\hbar}$$

which agrees with the earlier result.

For  $n = 1$ , the degeneracy is, from 0.13

$$(0.25) \quad d(1) = 3$$

The three possibilities are  $m = 0, \pm 1$  which are reflected in the three spherical harmonics  $Y_1^{0,\pm 1}$ . The radial function is the same in all cases, and is obtained from  $q_{max} = 0, l = 1$ . From 0.7, this gives

$$(0.26) \quad v(\rho) = c_0$$

$$(0.27) \quad u(\rho) = c_0 \rho^2 e^{-\rho_0 \rho^2/2}$$

$$(0.28) \quad \psi_{11m} = \frac{u}{r} Y_1^m$$

$$(0.29) \quad = c_0 \kappa^2 r e^{-\rho_0 \rho^2/2} Y_1^m$$

$$(0.30) \quad = c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} Y_1^m$$

Again, we work out  $c_0$  by imposing normalization. For example

$$(0.31) \quad \psi_{111} = c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} Y_1^1$$

$$(0.32) \quad = -c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$$

The normalization integral is

$$(0.33)$$

$$\int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_{111}^2 r^2 \sin\theta dr d\theta d\phi = c_0^2 \left(\frac{5\mu\omega}{\hbar}\right)^2 \frac{3}{8\pi} 2\pi \int_0^\pi \int_0^\infty e^{-\mu\omega r^2/\hbar} r^4 \sin^3\theta dr d\theta$$

$$(0.34) \quad = c_0^2 \frac{75}{8} \sqrt{\frac{\pi\hbar}{\mu\omega}} = 1$$

$$(0.35) \quad c_0 = \frac{2\sqrt{6}}{15} \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4}$$

I used Maple to do the integrals. This gives a wave function of

$$(0.36) \quad \psi_{111} = -\sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{i\phi}$$

We can work out the other two wave functions the same way (I used Maple, so I won't go into the details):

$$(0.37) \quad \psi_{11-1} = \sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{-i\phi}$$

$$(0.38) \quad \psi_{110} = \sqrt{\frac{2\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \cos\theta$$

The  $\psi_{110}$  here is the same as  $\psi_{001}$  in our rectangular solution set. The other two are linear combinations of  $\psi_{100}$  and  $\psi_{010}$  from our rectangular

set, which were (the suffixes in these 2 equations stand for  $x$ ,  $y$  and  $z$ , and not  $n$ ,  $l$  and  $m$ ):

$$(0.39) \quad \psi_{100} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \cos\phi$$

$$(0.40) \quad \psi_{010} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \sin\phi$$

We have

$$(0.41) \quad \psi_{111} = \frac{1}{\sqrt{2}} (\psi_{100} + i\psi_{010})$$

$$(0.42) \quad \psi_{11-1} = \frac{1}{\sqrt{2}} (\psi_{100} - i\psi_{010})$$