ISOTROPIC HARMONIC OSCILLATOR IN 3-D: USE OF SPHERICAL HARMONICS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've solved the 3-d isotropic harmonic oscillator before, so we've already solved most of Shankar's exercise 12.6.11. We can quote the results here. The solution has the form

$$\psi_{Elm}=\frac{U_{El}(r)}{r}Y_l^m(\theta,\phi) \tag{1}$$

The earlier solution uses notation from Griffiths's book, but as the end result is the same, it's not worth going through the derivation again using Shankar's notation.

The potential is

$$V(r)=\frac{1}{2}m\omega^2r^2 \tag{2}$$

The radial equation to be solved is

$$\frac{d^2u}{d\rho^2}=\left(-1+\frac{l(l+1)}{\rho^2}+\rho_0^2\rho^2\right)u \tag{3}$$

If we define

$$\kappa^2 \equiv \frac{2\mu E}{\hbar^2} \tag{4}$$

$$\rho \equiv \kappa r \tag{5}$$

$$\rho_0 \equiv \frac{\mu\omega}{\hbar\kappa^2}=\frac{\hbar\omega}{2E} \tag{6}$$

Taking the asymptotic behaviour of the radial function for small and large r into account leads us to a solution of form

$$u(\rho)=\rho^{l+1}e^{-\rho_0\rho^2/2}v(\rho) \tag{7}$$

Note that Griffiths's v is not the same as Shankar's v, the latter of which is defined by Shankar's equation 12.6.49.

This gives a differential equation for Griffiths's v

$$\rho\frac{d^2v}{d\rho^2}+2\left(l+1-\rho_0\rho^2\right)\frac{dv}{d\rho}+\rho(1-\rho_0(2l+3))v=0 \tag{8}$$

The function v can be solved as a power series, giving

$$v(\rho)=\sum c_j\rho^j \tag{9}$$

Substituting into 8 leads to the recursion relation

$$c_{q+2}=\frac{\rho_0(2q+2l+3)-1}{(q+2)(q+2l+3)}c_q \tag{10}$$

with $c_1=0$, so that $c_q=0$ for all odd q. The requirement that the series terminates at some finite value of j leads to the quantization condition on E:

$$E = \hbar\omega\left(q_{max}+l+\frac{3}{2}\right) \tag{11}$$

or, defining $n=q_{max}+l$,

$$E_n=\hbar\omega\left(n+\frac{3}{2}\right) \tag{12}$$

We worked out the degeneracies in the earlier post as well, so that the degeneracy of E_n is

$$d(n)=\frac{1}{2}(n+1)(n+2) \tag{13}$$

To complete Shankar's exercise, we need to work out the eigenfunctions for $n=0$ and $n=1$. For $n=0$, $q_{max}=l=0$, so only $c_0\neq0$ and we have

$$\begin{aligned}
v(\rho) &= c_0 && (14)\\
u(\rho) &= c_0\rho e^{-\rho_0\rho^2/2} && (15)\\
\psi_{000} &= \frac{u}{r}Y_0^0 && (16)\\
&= c_0\kappa e^{-\rho_0\rho^2/2}Y_0^0 && (17)\\
&= c_0\sqrt{\frac{2\mu3\omega}{4\pi\hbar}}e^{-\mu\omega r^2/2\hbar} && (18)
\end{aligned}$$

where in the fourth line we used

$$\kappa = \frac{\sqrt{2\mu E}}{\hbar} = \frac{\sqrt{2\mu\frac{3}{2}\hbar\omega}}{\hbar} = \sqrt{\frac{3\mu\omega}{\hbar}} \tag{19}$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \tag{20}$$

Normalizing this requires that

$$\begin{aligned} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\psi_{000}^2 r^2 \sin\theta dr\, d\theta\, d\phi &= c_0^2\frac{6\mu\omega}{\hbar}\int_0^{\infty}e^{-\mu\omega r^2/\hbar}r^2 dr & (21)\\ &= 1 & (22)\end{aligned}$$

This is a standard Gaussian integral and can be done using software or tables so we get

$$c_0 = \frac{\sqrt{6}}{3}\left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4} \tag{23}$$

This gives a wave function of

$$\psi_{000} = \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} e^{-\mu\omega r^2/2\hbar} \tag{24}$$

which agrees with the earlier result.
For $n = 1$, the degeneracy is, from 13

$$d(1) = 3 \tag{25}$$

The three possibilities are $m = 0, \pm 1$ which are reflected in the three spherical harmonics $Y_1^{0,\pm 1}$. The radial function is the same in all cases, and is obtained from $q_{max} = 0$, $l = 1$. From 7, this gives

$$\begin{aligned} v(\rho) &= c_0 & (26)\\ u(\rho) &= c_0\rho^2 e^{-\rho_0\rho^2/2} & (27)\\ \psi_{11m} &= \frac{u}{r}Y_1^m & (28)\\ &= c_0\kappa^2 r e^{-\rho_0\rho^2/2}Y_1^m & (29)\\ &= c_0\frac{5\mu\omega}{\hbar}re^{-\mu\omega r^2/2\hbar}Y_1^m & (30)\end{aligned}$$

Again, we work out c_0 by imposing normalization. For example

$$\psi_{111} = c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} Y_1^1 \tag{31}$$

$$= -c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{32}$$

The normalization integral is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi_{111}^2 r^2 \sin\theta dr\, d\theta\, d\phi = c_0^2 \left(\frac{5\mu\omega}{\hbar}\right)^2 \frac{3}{8\pi} 2\pi \int_0^{\pi}\int_0^{\infty} e^{-\mu\omega r^2/\hbar} r^4 \sin^3\theta dr\, d\theta \tag{33}$$

$$= c_0^2 \frac{75}{8} \sqrt{\frac{\pi\hbar}{\mu\omega}} = 1 \tag{34}$$

$$c_0 = \frac{2\sqrt{6}}{15} \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4} \tag{35}$$

I used Maple to do the integrals. This gives a wave function of

$$\psi_{111} = -\sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{i\phi} \tag{36}$$

We can work out the other two wave functions the same way (I used Maple, so I won't go into the details):

$$\psi_{11-1} = \sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{-i\phi} \tag{37}$$

$$\psi_{110} = \sqrt{\frac{2\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \cos\theta \tag{38}$$

The ψ_{110} here is the same as ψ_{001} in our rectangular solution set. The other two are linear combinations of ψ_{100} and ψ_{010} from our rectangular set, which were (the suffixes in these 2 equations stand for x, y and z, and not n, l and m):

$$\psi_{100} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \cos\phi \tag{39}$$

$$\psi_{010} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \sin\phi \tag{40}$$

We have

$$\psi_{111} = \frac{1}{\sqrt{2}}\left(\psi_{100}+i\psi_{010}\right) \tag{41}$$

$$\psi_{11-1} = \frac{1}{\sqrt{2}}\left(\psi_{100}-i\psi_{010}\right) \tag{42}$$