

ISOTROPIC HARMONIC OSCILLATOR IN 3-D: USE OF SPHERICAL HARMONICS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.11.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've solved the 3-d isotropic harmonic oscillator before, so we've already solved most of Shankar's exercise 12.6.11. We can quote the results here. The solution has the form

$$\psi_{Elm} = \frac{U_{El}(r)}{r} Y_l^m(\theta, \phi) \quad (1)$$

The earlier solution uses notation from Griffiths's book, but as the end result is the same, it's not worth going through the derivation again using Shankar's notation.

The potential is

$$V(r) = \frac{1}{2} m \omega^2 r^2 \quad (2)$$

The radial equation to be solved is

$$\frac{d^2 u}{d\rho^2} = \left(-1 + \frac{l(l+1)}{\rho^2} + \rho_0^2 \rho^2 \right) u \quad (3)$$

If we define

$$\kappa^2 \equiv \frac{2\mu E}{\hbar^2} \quad (4)$$

$$\rho \equiv \kappa r \quad (5)$$

$$\rho_0 \equiv \frac{\mu \omega}{\hbar \kappa^2} = \frac{\hbar \omega}{2E} \quad (6)$$

Taking the asymptotic behaviour of the radial function for small and large r into account leads us to a solution of form

$$u(\rho) = \rho^{l+1} e^{-\rho_0 \rho^2 / 2} v(\rho) \quad (7)$$

Note that Griffiths's v is not the same as Shankar's v , the latter of which is defined by Shankar's equation 12.6.49.

This gives a differential equation for Griffiths's v

$$\rho \frac{d^2 v}{d\rho^2} + 2(l+1 - \rho_0 \rho^2) \frac{dv}{d\rho} + \rho(1 - \rho_0(2l+3))v = 0 \quad (8)$$

The function v can be solved as a power series, giving

$$v(\rho) = \sum c_j \rho^j \quad (9)$$

Substituting into 8 leads to the recursion relation

$$c_{q+2} = \frac{\rho_0(2q+2l+3) - 1}{(q+2)(q+2l+3)} c_q \quad (10)$$

with $c_1 = 0$, so that $c_q = 0$ for all odd q . The requirement that the series terminates at some finite value of j leads to the quantization condition on E :

$$E = \hbar\omega \left(q_{max} + l + \frac{3}{2} \right) \quad (11)$$

or, defining $n = q_{max} + l$,

$$E_n = \hbar\omega \left(n + \frac{3}{2} \right) \quad (12)$$

We worked out the degeneracies in the earlier post as well, so that the degeneracy of E_n is

$$d(n) = \frac{1}{2}(n+1)(n+2) \quad (13)$$

To complete Shankar's exercise, we need to work out the eigenfunctions for $n = 0$ and $n = 1$. For $n = 0$, $q_{max} = l = 0$, so only $c_0 \neq 0$ and we have

$$v(\rho) = c_0 \quad (14)$$

$$u(\rho) = c_0 \rho e^{-\rho_0 \rho^2 / 2} \quad (15)$$

$$\psi_{000} = \frac{u}{r} Y_0^0 \quad (16)$$

$$= c_0 \kappa e^{-\rho_0 \rho^2 / 2} Y_0^0 \quad (17)$$

$$= c_0 \sqrt{\frac{2\mu 3\omega}{4\pi\hbar}} e^{-\mu\omega r^2 / 2\hbar} \quad (18)$$

where in the fourth line we used

$$\kappa = \frac{\sqrt{2\mu E}}{\hbar} = \frac{\sqrt{2\mu \frac{3}{2}\hbar\omega}}{\hbar} = \sqrt{\frac{3\mu\omega}{\hbar}} \quad (19)$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad (20)$$

Normalizing this requires that

$$\int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_{000}^2 r^2 \sin\theta dr d\theta d\phi = c_0^2 \frac{6\mu\omega}{\hbar} \int_0^\infty e^{-\mu\omega r^2/\hbar} r^2 dr \quad (21)$$

$$= 1 \quad (22)$$

This is a standard Gaussian integral and can be done using software or tables so we get

$$c_0 = \frac{\sqrt{6}}{3} \left(\frac{\mu\omega}{\pi\hbar} \right)^{1/4} \quad (23)$$

This gives a wave function of

$$\psi_{000} = \left(\frac{\mu\omega}{\pi\hbar} \right)^{3/4} e^{-\mu\omega r^2/2\hbar} \quad (24)$$

which agrees with the earlier result.

For $n = 1$, the degeneracy is, from 13

$$d(1) = 3 \quad (25)$$

The three possibilities are $m = 0, \pm 1$ which are reflected in the three spherical harmonics $Y_1^{0,\pm 1}$. The radial function is the same in all cases, and is obtained from $q_{max} = 0, l = 1$. From 7, this gives

$$v(\rho) = c_0 \quad (26)$$

$$u(\rho) = c_0 \rho^2 e^{-\rho_0 \rho^2/2} \quad (27)$$

$$\psi_{11m} = \frac{u}{r} Y_1^m \quad (28)$$

$$= c_0 \kappa^2 r e^{-\rho_0 \rho^2/2} Y_1^m \quad (29)$$

$$= c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} Y_1^m \quad (30)$$

Again, we work out c_0 by imposing normalization. For example

$$\psi_{111} = c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} Y_1^1 \quad (31)$$

$$= -c_0 \frac{5\mu\omega}{\hbar} r e^{-\mu\omega r^2/2\hbar} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \quad (32)$$

The normalization integral is

$$\int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_{111}^2 r^2 \sin\theta dr d\theta d\phi = c_0^2 \left(\frac{5\mu\omega}{\hbar}\right)^2 \frac{3}{8\pi} 2\pi \int_0^\pi \int_0^\infty e^{-\mu\omega r^2/\hbar} r^4 \sin^3\theta dr d\theta \quad (33)$$

$$= c_0^2 \frac{75}{8} \sqrt{\frac{\pi\hbar}{\mu\omega}} = 1 \quad (34)$$

$$c_0 = \frac{2\sqrt{6}}{15} \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4} \quad (35)$$

I used Maple to do the integrals. This gives a wave function of

$$\psi_{111} = -\sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{i\phi} \quad (36)$$

We can work out the other two wave functions the same way (I used Maple, so I won't go into the details):

$$\psi_{11-1} = \sqrt{\frac{\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \sin\theta e^{-i\phi} \quad (37)$$

$$\psi_{110} = \sqrt{\frac{2\mu\omega}{\hbar}} \left(\frac{\mu\omega}{\pi\hbar}\right)^{3/4} r e^{-\mu\omega r^2/2\hbar} \cos\theta \quad (38)$$

The ψ_{110} here is the same as ψ_{001} in our rectangular solution set. The other two are linear combinations of ψ_{100} and ψ_{010} from our rectangular set, which were (the suffixes in these 2 equations stand for x , y and z , and not n , l and m):

$$\psi_{100} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \cos\phi \quad (39)$$

$$\psi_{010} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin\theta \sin\phi \quad (40)$$

We have

$$\psi_{111} = \frac{1}{\sqrt{2}}(\psi_{100} + i\psi_{010}) \quad (41)$$

$$\psi_{11-1} = \frac{1}{\sqrt{2}}(\psi_{100} - i\psi_{010}) \quad (42)$$