HYDROGEN ATOM: RADIAL FUNCTION AT LARGE R

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 13, Exercise 13.1.4.

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When solving the 3-d Schrödinger equation for a spherically symmetric potential, the radial function has the asymptotic form for large r, and for the energy $E<0$:

$$U_{El}(r) \underset{r\to\infty}{\longrightarrow} Ar^{\pm me^2/\kappa\hbar^2}e^{-\kappa r} \tag{1}$$

where

$$\kappa \equiv \sqrt{\frac{2m|E|}{\hbar^2}} \tag{2}$$

For the hydrogen atom, the function U_{El} is obtained from a series solution of the differential equation with the result

$$U_{El} = e^{-\rho}v_{El} \tag{3}$$

$$v_{El} = \rho^{l+1}\sum_{k=0}^{\infty} C_k\rho^k \tag{4}$$

$$\rho = \sqrt{\frac{-2mE}{\hbar^2}}r \tag{5}$$

$$= \kappa r \tag{6}$$

To keep the wave function finite at large r, we require the series to terminate, which leads to the quantized energy levels, given by

$$E_n = -\frac{me^4}{2\hbar^2 n^2} \tag{7}$$

The series in 4 terminates at a value of $k=n-l-1$, so the function v_{El} is a polynomial in ρ, and thus in r, of degree n. Since the actual radial function is

$$R_{nl} = \frac{U_{El}}{r} \tag{8}$$

we have that R_{nl} is a polynomial of degree $n-1$ in r multiplied by the exponential $e^{-\rho} = e^{-\kappa r}$. That is, for large r

$$R_{nl} \sim r^{n-1} e^{-\kappa r} \tag{9}$$

To show that this is consistent with 1, we use 7 and 2.

$$\begin{aligned} n &= \sqrt{\frac{me^4}{2\hbar^2 |E_n|}} && (10) \\ &= \sqrt{\frac{me^4}{2\hbar^2}} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{\kappa} && (11) \\ &= \frac{me^2}{\kappa \hbar^2} && (12) \end{aligned}$$

Comparing this with 1, we see that

$$r^n = r^{me^2/\kappa\hbar^2} \tag{13}$$

so the condition is satisfied.