

## HYDROGEN ATOM: RADIAL FUNCTION AT LARGE R

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 13, Exercise 13.1.4.

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When solving the 3-d Schrödinger equation for a spherically symmetric potential, the radial function has the asymptotic form for large  $r$ , and for the energy  $E < 0$ :

$$U_{El}(r) \xrightarrow{r \rightarrow \infty} A r^{\pm me^2 / \kappa \hbar^2} e^{-\kappa r} \quad (1)$$

where

$$\kappa \equiv \sqrt{\frac{2m|E|}{\hbar^2}} \quad (2)$$

For the hydrogen atom, the function  $U_{El}$  is obtained from a series solution of the differential equation with the result

$$U_{El} = e^{-\rho} v_{El} \quad (3)$$

$$v_{El} = \rho^{l+1} \sum_{k=0}^{\infty} C_k \rho^k \quad (4)$$

$$\rho = \sqrt{\frac{-2mE}{\hbar^2}} r \quad (5)$$

$$= \kappa r \quad (6)$$

To keep the wave function finite at large  $r$ , we require the series to terminate, which leads to the quantized energy levels, given by

$$E_n = -\frac{me^4}{2\hbar^2 n^2} \quad (7)$$

The series in 4 terminates at a value of  $k = n - l - 1$ , so the function  $v_{El}$  is a polynomial in  $\rho$ , and thus in  $r$ , of degree  $n$ . Since the actual radial function is

$$R_{nl} = \frac{U_{El}}{r} \quad (8)$$

we have that  $R_{nl}$  is a polynomial of degree  $n - 1$  in  $r$  multiplied by the exponential  $e^{-\rho} = e^{-\kappa r}$ . That is, for large  $r$

$$R_{nl} \sim r^{n-1} e^{-\kappa r} \quad (9)$$

To show that this is consistent with 1, we use 7 and 2.

$$n = \sqrt{\frac{me^4}{2\hbar^2 |E_n|}} \quad (10)$$

$$= \sqrt{\frac{me^4}{2\hbar^2}} \sqrt{\frac{2m}{\hbar^2} \frac{1}{\kappa}} \quad (11)$$

$$= \frac{me^2}{\kappa\hbar^2} \quad (12)$$

Comparing this with 1, we see that

$$r^n = r^{me^2/\kappa\hbar^2} \quad (13)$$

so the condition is satisfied.