

HYDROGEN ATOM: RADIAL FUNCTION AT LARGE R

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 13, Exercise 13.1.4.

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When solving the 3-d Schrödinger equation for a spherically symmetric potential, the radial function has the asymptotic form for large r , and for the energy $E < 0$:

$$(0.1) \quad U_{El}(r) \xrightarrow{r \rightarrow \infty} A r^{\pm me^2 / \kappa \hbar^2} e^{-\kappa r}$$

where

$$(0.2) \quad \kappa \equiv \sqrt{\frac{2m|E|}{\hbar^2}}$$

For the hydrogen atom, the function U_{El} is obtained from a series solution of the differential equation with the result

$$(0.3) \quad U_{El} = e^{-\rho} v_{El}$$

$$(0.4) \quad v_{El} = \rho^{l+1} \sum_{k=0}^{\infty} C_k \rho^k$$

$$(0.5) \quad \rho = \sqrt{\frac{-2mE}{\hbar^2}} r$$

$$(0.6) \quad = \kappa r$$

To keep the wave function finite at large r , we require the series to terminate, which leads to the quantized energy levels, given by

$$(0.7) \quad E_n = -\frac{me^4}{2\hbar^2 n^2}$$

The series in 0.4 terminates at a value of $k = n - l - 1$, so the function v_{El} is a polynomial in ρ , and thus in r , of degree n . Since the actual radial function is

$$(0.8) \quad R_{nl} = \frac{U_{El}}{r}$$

we have that R_{nl} is a polynomial of degree $n - 1$ in r multiplied by the exponential $e^{-\rho} = e^{-\kappa r}$. That is, for large r

$$(0.9) \quad R_{nl} \sim r^{n-1} e^{-\kappa r}$$

To show that this is consistent with 0.1, we use 0.7 and 0.2.

$$(0.10) \quad n = \sqrt{\frac{me^4}{2\hbar^2 |E_n|}}$$

$$(0.11) \quad = \sqrt{\frac{me^4}{2\hbar^2}} \sqrt{\frac{2m}{\hbar^2} \frac{1}{\kappa}}$$

$$(0.12) \quad = \frac{me^2}{\kappa\hbar^2}$$

Comparing this with 0.1, we see that

$$(0.13) \quad r^n = r^{me^2/\kappa\hbar^2}$$

so the condition is satisfied.