

RUNGE LENZ VECTOR AND CLOSED ORBITS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 13, Exercise 13.2.1.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The energy levels of hydrogen, when calculated from the Coulomb potential alone (ignoring various perturbations) depend only on the principal quantum number n according to

$$(1) \quad E = -\frac{1}{n^2} \frac{\mu e^4}{2\hbar^2}$$

The quantization arises entirely from the requirement that the radial function remain finite for large r , and makes no mention of the angular quantum numbers l and m . Thus each energy level (each value of n) has a degeneracy of n^2 , with $2l + 1$ degenerate states for each l . Each symmetry is associated with the conservation of some dynamical quantity, with the degeneracy in m due to conservation of angular momentum.

Shankar points out that, in classical mechanics, any potential with a $\frac{1}{r}$ dependence conserves the Runge-Lenz vector, defined for the hydrogen atom potential as

$$(2) \quad \mathbf{n} = \frac{\mathbf{p} \times \ell}{\mu} - \frac{e^2}{r} \mathbf{r}$$

where I've used μ for the electron mass to avoid confusion with the L_z quantum number m .

Although it doesn't make sense to talk about the orbit of the electron in quantum mechanics, classically we can see that the conservation of \mathbf{n} implies that the orbit is closed. We can see this as follows.

First, using

$$(3) \quad \ell = \mathbf{r} \times \mathbf{p}$$

we have

$$\begin{aligned}
 (4) \quad \mathbf{n} &= \frac{1}{\mu} \mathbf{p} \times (\mathbf{r} \times \mathbf{p}) - \frac{e^2}{r} \mathbf{r} \\
 (5) \quad &= \frac{1}{\mu} \mathbf{r} \times (\mathbf{p} \cdot \mathbf{p}) - \mathbf{p} (\mathbf{r} \cdot \mathbf{p}) - \frac{e^2}{r} \mathbf{r} \\
 (6) \quad &= \left(\frac{p^2}{\mu} - \frac{e^2}{r} \right) \mathbf{r} - \mathbf{p} (\mathbf{r} \cdot \mathbf{p})
 \end{aligned}$$

In the second line, we used the vector identity

$$(7) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Since we're dealing with a bound state, r must always remain finite, so it must have a maximum value. At this point $\frac{dr}{dt} = 0$, which means that there is no radial motion, which in turn means that all motion at that point must be perpendicular to \mathbf{r} . In other words

$$(8) \quad \mathbf{p} \cdot \mathbf{r}_{max} = 0$$

Also, from conservation of energy, we have

$$(9) \quad E = \frac{p^2}{2\mu} - \frac{e^2}{r}$$

so at r_{max} we have

$$(10) \quad p^2 = 2\mu \left(E + \frac{e^2}{r_{max}} \right)$$

Plugging these into 6, we get

$$(11) \quad \mathbf{n} = \left(2E + \frac{2e^2}{r_{max}} - \frac{e^2}{r_{max}} \right) \mathbf{r}_{max}$$

$$(12) \quad = \left(2E + \frac{e^2}{r_{max}} \right) \mathbf{r}_{max}$$

Exactly the same argument applies to the case where r is a minimum: again $\frac{dr}{dt} = 0$ so $\mathbf{r} \cdot \mathbf{p} = 0$ and we end up with

$$(13) \quad \mathbf{n} = \left(2E + \frac{e^2}{r_{min}} \right) \mathbf{r}_{min}$$

If \mathbf{n} is conserved (constant), then it must be parallel or anti-parallel to both \mathbf{r}_{max} and \mathbf{r}_{min} , and the latter two vectors must therefore always have the same direction. In other words, the particle reaches its maximum (and minimum) distance always at the same point in its orbit, meaning that the orbit is closed.

In a general (elliptical) orbit, $r_{max} > r_{min}$ so $\frac{e^2}{r_{max}} < \frac{e^2}{r_{min}}$. Since $E < 0$ for a bound orbit, we therefore must have

$$(14) \quad 2E + \frac{e^2}{r_{max}} < 0$$

$$(15) \quad 2E + \frac{e^2}{r_{min}} > 0$$

This in turn implies that \mathbf{n} is anti-parallel to \mathbf{r}_{max} and parallel to \mathbf{r}_{min} .

For a circular orbit, both r and p are constant, so both the kinetic and potential energies are also constant. From the virial theorem, we know that, for $V \propto \frac{1}{r}$

$$(16) \quad \langle T \rangle = -\frac{1}{2} \langle V \rangle$$

Thus

$$(17) \quad E = T + V$$

$$(18) \quad = \frac{V}{2}$$

$$(19) \quad = -\frac{e^2}{2r}$$

Thus from 12, we see that $\mathbf{n} = 0$ for a circular orbit.