SIZES OF ELEMENTARY PARTICLES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 13, Exercises 13.3.1 - 13.3.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Due to the position-momentum uncertainty principle, if we wish to determine the location of a particle to within a distance ΔX, the momentum of the photon used to detect the particle must satisfy

$$\Delta P \Delta X \geq \frac{\hbar}{2} \tag{1}$$

This relation is valid in non-relativistic quantum mechanics, where we are using position eigenkets $|X\rangle$ which define a particle's position exactly. To do this, however, would require a photon of infinite energy. In relativistic quantum theory, if the energy of the photon is large enough, it is possible to convert the energy into mass by creating a particle-antiparticle pair. If we're trying to determine the location of an electron, then if the energy of the bombarding photon is around twice the rest energy of an electron, this pair creation process can occur. Thus for practical purposes, the maximum photon energy that we can use to detect the electron is finite, which means that the electron's position can be determined only approximately.

To get an idea of the 'radius' of an electron using these ideas (I put 'radius' in quotes because an electron doesn't have a rigid boundary in quantum theory), we can proceed as follows. We'll work only to orders of magnitude, rather than precise quantities.

From the uncertaintly relation, the photon's momentum is about

$$\Delta P \sim \frac{\hbar}{\Delta X} \tag{2}$$

For a photon, the relativistic energy is related to the momentum by

$$\Delta E = \Delta P c \tag{3}$$

where c is the speed of light. Therefore, the energy of the incident photon is

$$\Delta E \sim \frac{\hbar c}{\Delta X} \tag{4}$$

We therefore want to restrict this energy to less than twice the electron's rest energy, so

$$\Delta E \lesssim 2mc^2 \tag{5}$$

which leads to

$$\frac{\hbar c}{\Delta X} \lesssim 2mc^2 \tag{6}$$

$$\Delta X \gtrsim \frac{\hbar}{2mc} \sim \frac{\hbar}{mc} \tag{7}$$

The latter quantity is the Compton wavelength of the electron. [When we originally encountered the Compton wavelength is Carroll & Ostlie's book on astrophysics, they defined it as h/mc, so Shankar's Compton wavelength is $\frac{1}{2\pi}$ times that of Carroll & Ostlie. However, since we're working with orders of magnitude, this won't matter much.]

Thus the Compton wavelength can be taken as a rough size of the electron. We can write this as a fraction of the Bohr radius a_0 using

$$a_0 \equiv \frac{\hbar^2}{me^2} \tag{8}$$

so that

$$\frac{\hbar/mc}{a_0} = \frac{\hbar}{mc}\frac{me^2}{\hbar^2} = \frac{e^2}{\hbar c} = \alpha \approx \frac{1}{137} \tag{9}$$

where α is the famous fine structure constant. Since a_0 is roughly the radius of a ground-state hydrogen atom, the electron is about 100 times smaller than this.

We can use similar arguments to do some rough calculations on other particles.

Example 1. For example, the pion has a range of about 10^{-15} m as a mediator of the nuclear force, so if we take this as ΔX then

$$2m_\pi c^2 \sim \frac{\hbar c}{\Delta X} \tag{10}$$

The rest energy of an electron is about 0.5 MeV, so we can get an estimate of the rest energy of the pion as follows.

$$\frac{m_\pi c^2}{m_e c^2} = \frac{\Delta X_e}{\Delta X_\pi} = \frac{a_0/137}{10^{-15}} \tag{11}$$

The Bohr radius is about

$$a_0 \approx 5 \times 10^{-11} \text{ m} \tag{12}$$

so

$$m_\pi c^2 \approx (0.5 \text{ MeV}) \frac{5 \times 10^{-11}}{137 \times 10^{-15}} = 182 \text{ MeV} \tag{13}$$

The actual rest mass of a pion is around 140 MeV, so this estimate isn't too bad.

Example 2. The de Broglie wavelength of a particle is defined by

$$\lambda = \frac{h}{p} \tag{14}$$

For an electron with kinetic energy 200 eV, we need to find its momentum to calculate λ. The relativistic kinetic energy is

$$K = mc^2 (\gamma - 1) \tag{15}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{16}$$

We have

$$\gamma = \frac{K}{mc^2} + 1 = \frac{200 \text{ eV}}{0.5 \times 10^6 \text{ eV}} + 1 = 1.0004 \tag{17}$$

Thus the electron is travelling at a non-relativistic speed, so to a good approximation we can use Newtonian formulas. The speed is

$$v = c\sqrt{\frac{2K}{mc^2}} = c\sqrt{\frac{2(200)}{0.5 \times 10^6}} \approx 0.03c \tag{18}$$

$$p = mv = \left(9.1 \times 10^{-31}\right)(0.03)\left(3 \times 10^8\right) = 7.7 \times 10^{-24} \text{ kg m s}^{-1} \tag{19}$$

$$\lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34}}{7.7 \times 10^{-24}} \approx 10^{-10} \text{ m} = 1 \tag{20}$$