

EVERY SPIN-1/2 SPINOR IS AN EIGENKET OF SOME SPIN OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.1.

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The eigenvectors of the spin $\frac{1}{2}$ matrix in an arbitrary direction are given by

$$(0.1) \quad |\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

$$(0.2) \quad |\hat{n}-\rangle = \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

where the direction vector is given by

$$(0.3) \quad \hat{\mathbf{n}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

The corresponding spin operator is given by the matrix

$$(0.4) \quad \hat{\mathbf{n}} \cdot \mathbf{S} = \frac{\hbar}{2} \begin{bmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{bmatrix}$$

Any 2-component normalized spinor is an eigenvector of such a matrix. To see this, suppose we have an arbitrary spinor written as

$$(0.5) \quad |\chi\rangle = \rho_1 e^{i\phi_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \rho_2 e^{i\phi_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(0.6) \quad = \begin{bmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{bmatrix}$$

where $\rho_{1,2}$ and $\phi_{1,2}$ are arbitrary real numbers (so that the coefficients on the RHS are arbitrary complex numbers). From normalization we have

$$(0.7) \quad \langle \chi | \chi \rangle = 1 = \begin{bmatrix} \rho_1 e^{-i\phi_1} & \rho_2 e^{-i\phi_2} \end{bmatrix} \begin{bmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{bmatrix} = \rho_1^2 + \rho_2^2$$

Thus we can write ρ_1 and ρ_2 as the sine and cosine of some angle, which we'll call $\frac{\theta}{2}$, giving

$$(0.8) \quad |\chi\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi_1} \\ \sin \frac{\theta}{2} e^{i\phi_2} \end{bmatrix}$$

We can put this in the form 0.1 as follows. Since an overall phase doesn't affect the physics of the spinor, we can write

$$(0.9) \quad |\chi\rangle = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi_1} \\ \sin \frac{\theta}{2} e^{i\phi_2} \end{bmatrix}$$

We have the conditions

$$(0.10) \quad \phi_1 = \alpha - \frac{\phi}{2}$$

$$(0.11) \quad \phi_2 = \alpha + \frac{\phi}{2}$$

Solving, we get

$$(0.12) \quad \alpha = \frac{\phi_1 + \phi_2}{2}$$

$$(0.13) \quad \phi = \phi_2 - \phi_1$$

giving

$$(0.14) \quad |\chi\rangle = e^{i(\phi_1 + \phi_2)/2} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i(\phi_2 - \phi_1)/2} \\ \sin \frac{\theta}{2} e^{i(\phi_2 - \phi_1)/2} \end{bmatrix}$$

Thus $|\chi\rangle$ as given by 0.6 is an eigenvector of the operator 0.4, where

$$(0.15) \quad \hat{\mathbf{n}} = \sin \theta \cos(\phi_2 - \phi_1) \hat{\mathbf{x}} + \sin \theta \sin(\phi_2 - \phi_1) \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$