EVERY SPIN-1/2 SPINOR IS AN EIGENKET OF SOME SPIN OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.1.

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The eigenvectors of the spin $\frac{1}{2}$ matrix in an arbitrary direction are given by

$$|\hat{n}+\rangle = \left[\begin{array}{c} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{array}\right] \quad (1)$$

$$|\hat{n}-\rangle = \left[\begin{array}{c} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{array}\right] \quad (2)$$

where the direction vector is given by

$$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \quad (3)$$

The corresponding spin operator is given by the matrix

$$\hat{\mathbf{n}}\cdot\mathbf{S} = \frac{\hbar}{2}\left[\begin{array}{cc} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{array}\right] \quad (4)$$

Any 2-component normalized spinor is an eigenvector of such a matrix. To see this, suppose we have an arbitrary spinor written as

$$|\chi\rangle = \rho_1 e^{i\phi_1}\left[\begin{array}{c} 1 \\ 0 \end{array}\right] + \rho_2 e^{i\phi_2}\left[\begin{array}{c} 0 \\ 1 \end{array}\right] \quad (5)$$

$$= \left[\begin{array}{c} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{array}\right] \quad (6)$$

where $\rho_{1,2}$ and $\phi_{1,2}$ are arbitrary real numbers (so that the coefficients on the RHS are arbitrary complex numbers). From normalization we have

$$\langle\chi|\chi\rangle = 1 = \left[\begin{array}{cc} \rho_1 e^{-i\phi_1} & \rho_2 e^{-i\phi_2} \end{array}\right]\left[\begin{array}{c} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{array}\right] = \rho_1^2 + \rho_2^2 \quad (7)$$

Thus we can write ρ_1 and ρ_2 as the sine and cosine of some angle, which we'll call $\frac{\theta}{2}$, giving

$$|\chi\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi_1} \\ \sin\frac{\theta}{2}e^{i\phi_2} \end{bmatrix} \tag{8}$$

We can put this in the form 1 as follows. Since an overall phase doesn't affect the physics of the spinor, we can write

$$|\chi\rangle = e^{i\alpha}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi_1} \\ \sin\frac{\theta}{2}e^{i\phi_2} \end{bmatrix} \tag{9}$$

We have the conditions

$$\phi_1 = \alpha - \frac{\phi}{2} \tag{10}$$

$$\phi_2 = \alpha + \frac{\phi}{2} \tag{11}$$

Solving, we get

$$\alpha = \frac{\phi_1+\phi_2}{2} \tag{12}$$

$$\phi = \phi_2 - \phi_1 \tag{13}$$

giving

$$|\chi\rangle = e^{i(\phi_1+\phi_2)/2}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i(\phi_2-\phi_1)/2} \\ \sin\frac{\theta}{2}e^{i(\phi_2-\phi_1)/2} \end{bmatrix} \tag{14}$$

Thus $|\chi\rangle$ as given by 6 is an eigenvector of the operator 4, where

$$\hat{\mathbf{n}} = \sin\theta\cos(\phi_2-\phi_1)\,\hat{\mathbf{x}} + \sin\theta\sin(\phi_2-\phi_1)\,\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{15}$$