

## EVERY SPIN-1/2 SPINOR IS AN EIGENKET OF SOME SPIN OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.1.

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The eigenvectors of the spin  $\frac{1}{2}$  matrix in an arbitrary direction are given by

$$|\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (1)$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (2)$$

where the direction vector is given by

$$\hat{\mathbf{n}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \quad (3)$$

The corresponding spin operator is given by the matrix

$$\hat{\mathbf{n}} \cdot \mathbf{S} = \frac{\hbar}{2} \begin{bmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{bmatrix} \quad (4)$$

Any 2-component normalized spinor is an eigenvector of such a matrix. To see this, suppose we have an arbitrary spinor written as

$$|\chi\rangle = \rho_1 e^{i\phi_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \rho_2 e^{i\phi_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{bmatrix} \quad (6)$$

where  $\rho_{1,2}$  and  $\phi_{1,2}$  are arbitrary real numbers (so that the coefficients on the RHS are arbitrary complex numbers). From normalization we have

$$\langle \chi | \chi \rangle = 1 = \begin{bmatrix} \rho_1 e^{-i\phi_1} & \rho_2 e^{-i\phi_2} \end{bmatrix} \begin{bmatrix} \rho_1 e^{i\phi_1} \\ \rho_2 e^{i\phi_2} \end{bmatrix} = \rho_1^2 + \rho_2^2 \quad (7)$$

Thus we can write  $\rho_1$  and  $\rho_2$  as the sine and cosine of some angle, which we'll call  $\frac{\theta}{2}$ , giving

$$|\chi\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi_1} \\ \sin \frac{\theta}{2} e^{i\phi_2} \end{bmatrix} \quad (8)$$

We can put this in the form 1 as follows. Since an overall phase doesn't affect the physics of the spinor, we can write

$$|\chi\rangle = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi_1} \\ \sin \frac{\theta}{2} e^{i\phi_2} \end{bmatrix} \quad (9)$$

We have the conditions

$$\phi_1 = \alpha - \frac{\phi}{2} \quad (10)$$

$$\phi_2 = \alpha + \frac{\phi}{2} \quad (11)$$

Solving, we get

$$\alpha = \frac{\phi_1 + \phi_2}{2} \quad (12)$$

$$\phi = \phi_2 - \phi_1 \quad (13)$$

giving

$$|\chi\rangle = e^{i(\phi_1 + \phi_2)/2} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i(\phi_2 - \phi_1)/2} \\ \sin \frac{\theta}{2} e^{i(\phi_2 - \phi_1)/2} \end{bmatrix} \quad (14)$$

Thus  $|\chi\rangle$  as given by 6 is an eigenvector of the operator 4, where

$$\hat{\mathbf{n}} = \sin \theta \cos (\phi_2 - \phi_1) \hat{\mathbf{x}} + \sin \theta \sin (\phi_2 - \phi_1) \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \quad (15)$$