# PAULI MATRICES: PROPERTIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.3.

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The three components of the spin operator $\mathbf{S}$ for spin $\frac{1}{2}$ can be expressed in terms of the Pauli matrices

$$\sigma_x = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]; \quad \sigma_y = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]; \quad \sigma_z = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] \tag{1}$$

as

$$S_i = \frac{\hbar}{2}\sigma_i \tag{2}$$

As the trace of a matrix is the sum of its diagonal elements, it's obvious from their definitions that the $\sigma_i$ are traceless, but for some reason Shankar wants us to show this by a roundabout method.

We can show by direct calculation that the Pauli matrices anticommute with each other. For example

$$\begin{aligned}
\sigma_x\sigma_y &= \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right] && (3) \\
&= \left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right] && (4) \\
&= -\left[\begin{array}{cc} -i & 0 \\ 0 & i \end{array}\right] && (5) \\
&= -\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] && (6) \\
&= -\sigma_y\sigma_x && (7)
\end{aligned}$$

In general, we have, for $i \neq j$:

$$\left[\sigma_i,\sigma_j\right]_+ = 0 \tag{8}$$
$$\sigma_i\sigma_j = -\sigma_j\sigma_i \tag{9}$$

Also, by direct calculation (or by using the commutation relations for $S_i$) we can show that

$$\left[\sigma_x,\sigma_y\right] = \sigma_x\sigma_y-\sigma_y\sigma_x \tag{10}$$
$$= 2\sigma_x\sigma_y \tag{11}$$
$$= 2\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{12}$$
$$= 2i\sigma_z \tag{13}$$

This gives the relation

$$\sigma_x\sigma_y = i\sigma_z \tag{14}$$

and also for cyclic permutations of $x$, $y$ and $z$. Also by direct calculation we can see that

$$\sigma_i^2 = I \tag{15}$$

We can write this more generally as

$$\sigma_i\sigma_j = \delta_{ij}I + i\sum_k \varepsilon_{ijk}\sigma_k \tag{16}$$

where $\varepsilon_{ijk}$ is the Levi-Civita antisymmetric tensor.

Returning to the trace, we can use the theorem for the trace of a product:

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA) \tag{17}$$

Applying this to 9 we have

$$\mathrm{Tr}\,(\sigma_x\sigma_y) = \mathrm{Tr}\,(\sigma_y\sigma_x) = -\mathrm{Tr}\,(\sigma_y\sigma_x) \tag{18}$$

Any quantity equal to its negative must be zero, so

$$\mathrm{Tr}\,(\sigma_x\sigma_y) = 0 \tag{19}$$

Thus from 14 we get

$$\mathrm{Tr}\sigma_z = 0 \tag{20}$$

We can use the same argument for $\sigma_x$ and $\sigma_y$ by cyclic permutation.

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