

PAULI MATRICES: PROPERTIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.3.

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The three components of the spin operator \mathbf{S} for spin $\frac{1}{2}$ can be expressed in terms of the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

as

$$S_i = \frac{\hbar}{2} \sigma_i \quad (2)$$

As the trace of a matrix is the sum of its diagonal elements, it's obvious from their definitions that the σ_i are traceless, but for some reason Shankar wants us to show this by a roundabout method.

We can show by direct calculation that the Pauli matrices anticommute with each other. For example

$$\sigma_x \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad (4)$$

$$= - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad (5)$$

$$= - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (6)$$

$$= -\sigma_y \sigma_x \quad (7)$$

In general, we have, for $i \neq j$:

$$[\sigma_i, \sigma_j]_+ = 0 \quad (8)$$

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \quad (9)$$

Also, by direct calculation (or by using the commutation relations for S_i) we can show that

$$[\sigma_x, \sigma_y] = \sigma_x \sigma_y - \sigma_y \sigma_x \quad (10)$$

$$= 2\sigma_x \sigma_y \quad (11)$$

$$= 2 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad (12)$$

$$= 2i\sigma_z \quad (13)$$

This gives the relation

$$\sigma_x \sigma_y = i\sigma_z \quad (14)$$

and also for cyclic permutations of x , y and z . Also by direct calculation we can see that

$$\sigma_i^2 = I \quad (15)$$

We can write this more generally as

$$\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \varepsilon_{ijk} \sigma_k \quad (16)$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor.

Returning to the trace, we can use the theorem for the trace of a product:

$$\text{Tr}(AB) = \text{Tr}(BA) \quad (17)$$

Applying this to 9 we have

$$\text{Tr}(\sigma_x \sigma_y) = \text{Tr}(\sigma_y \sigma_x) = -\text{Tr}(\sigma_y \sigma_x) \quad (18)$$

Any quantity equal to its negative must be zero, so

$$\text{Tr}(\sigma_x \sigma_y) = 0 \quad (19)$$

Thus from 14 we get

$$\text{Tr}\sigma_z = 0 \quad (20)$$

We can use the same argument for σ_x and σ_y by cyclic permutation.

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