

PAULI MATRICES: PROPERTIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.3.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The three components of the spin operator \mathbf{S} for spin $\frac{1}{2}$ can be expressed in terms of the Pauli matrices

$$(0.1) \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

as

$$(0.2) \quad S_i = \frac{\hbar}{2} \sigma_i$$

As the trace of a matrix is the sum of its diagonal elements, it's obvious from their definitions that the σ_i are traceless, but for some reason Shankar wants us to show this by a roundabout method.

We can show by direct calculation that the Pauli matrices anticommute with each other. For example

$$(0.3) \quad \sigma_x \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$(0.4) \quad = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$(0.5) \quad = - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$(0.6) \quad = - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(0.7) \quad = -\sigma_y \sigma_x$$

In general, we have, for $i \neq j$:

$$(0.8) \quad [\sigma_i, \sigma_j]_+ = 0$$

$$(0.9) \quad \sigma_i \sigma_j = -\sigma_j \sigma_i$$

Also, by direct calculation (or by using the commutation relations for S_i) we can show that

$$(0.10) \quad [\sigma_x, \sigma_y] = \sigma_x \sigma_y - \sigma_y \sigma_x$$

$$(0.11) \quad = 2\sigma_x \sigma_y$$

$$(0.12) \quad = 2 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$(0.13) \quad = 2i\sigma_z$$

This gives the relation

$$(0.14) \quad \sigma_x \sigma_y = i\sigma_z$$

and also for cyclic permutations of x , y and z . Also by direct calculation we can see that

$$(0.15) \quad \sigma_i^2 = I$$

We can write this more generally as

$$(0.16) \quad \sigma_i \sigma_j = \delta_{ij} I + i \sum_k \varepsilon_{ijk} \sigma_k$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor.

Returning to the trace, we can use the theorem for the trace of a product:

$$(0.17) \quad \text{Tr}(AB) = \text{Tr}(BA)$$

Applying this to 0.9 we have

$$(0.18) \quad \text{Tr}(\sigma_x \sigma_y) = \text{Tr}(\sigma_y \sigma_x) = -\text{Tr}(\sigma_y \sigma_x)$$

Any quantity equal to its negative must be zero, so

$$(0.19) \quad \text{Tr}(\sigma_x \sigma_y) = 0$$

Thus from 0.14 we get

$$(0.20) \quad \text{Tr}\sigma_z = 0$$

We can use the same argument for σ_x and σ_y by cyclic permutation.

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