PAULI MATRICES: A USEFUL IDENTITY

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The three components of the spin operator **S** for spin $\frac{1}{2}$ can be expressed in terms of the Pauli matrices

$$\sigma_x = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]; \quad \sigma_y = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]; \quad \sigma_z = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] \tag{1}$$

We can derive an identity involving the Pauli matrices:

$$(\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma) = (\mathbf{A}\cdot\mathbf{B})\,I + i\,(\mathbf{A}\times\mathbf{B})\cdot\sigma \tag{2}$$

One way of proving this is to use the commutation relations for the Pauli matrices. We have

$$[\sigma_i,\sigma_j]_+ = 2\delta_{ij}I \tag{3}$$

$$[\sigma_i,\sigma_j] = 2i\sum_k \varepsilon_{ijk}\sigma_k \tag{4}$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor.

We therefore have

$$\sigma_i\sigma_j = \frac{1}{2}\left([\sigma_i,\sigma_j]_+ + [\sigma_i,\sigma_j]\right) \tag{5}$$

$$= \delta_{ij}I + i\sum_k \varepsilon_{ijk}\sigma_k \tag{6}$$

Using the summation convention where repeated indices are summed from 1 to 3 (that is, over x, y and z):

$$
\begin{aligned}
(\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma) &= A_i\sigma_i B_j\sigma_j && (7)\\
&= A_i B_j \sigma_i\sigma_j && (8)\\
&= A_i B_j\left(\delta_{ij}I + i\varepsilon_{ijk}\sigma_k\right) && (9)\\
&= A_i B_i I + i\varepsilon_{ijk}A_i B_j\sigma_k && (10)\\
&= (\mathbf{A}\cdot\mathbf{B})\,I + i(\mathbf{A}\times\mathbf{B})\cdot\sigma && (11)
\end{aligned}
$$

where the last term on the RHS follows from writing the vector cross product in terms of ε_{ijk}. [Note that in the second line, we've assumed that **B** commutes with σ.]

Another way of deriving this result is as follows. First, we add the 2×2 identity matrix I to the set of Pauli matrices, calling it $\sigma_0 \equiv I$. Then, because we have four independent matrices (Shankar shows they are linearly independent in his equations 14.3.40-41) each with 4 entries, we can write any 2×2 complex matrix as a linear combination of the σ_α (where a Greek subscript ranges from 0 to 3). That is, for a general 2×2 matrix M

$$M = \sum_\alpha m_\alpha \sigma_\alpha \tag{12}$$

From the trace identities

$$\mathrm{Tr}\left(\sigma_\alpha\sigma_\beta\right) = 2\delta_{\alpha\beta} \tag{13}$$

we can find m_α by right-multiplying by σ_β and taking the trace:

$$
\begin{aligned}
\mathrm{Tr}\left(M\sigma_\beta\right) &= \sum_\alpha m_\alpha \mathrm{Tr}\left(\sigma_\alpha\sigma_\beta\right) && (14)\\
&= 2\sum_\alpha m_\alpha\delta_{\alpha\beta} && (15)\\
&= 2m_\beta && (16)
\end{aligned}
$$

Thus

$$m_\alpha = \frac{1}{2}\mathrm{Tr}\left(M\sigma_\alpha\right) \tag{17}$$

Returning to 2, we can identify (again using the summation convention):

$$
\begin{aligned}
M &= (\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma) && (18)\\
&= A_i\sigma_i B_j\sigma_j && (19)\\
&= m_\alpha\sigma_\alpha && (20)
\end{aligned}
$$

For $\alpha = 0$ we have

$$m_0 = \frac{1}{2}\mathrm{Tr}\left(M\sigma_0\right) \tag{21}$$
$$= \frac{1}{2}\mathrm{Tr}\left(M\right) \tag{22}$$
$$= \frac{1}{2}A_iB_j\mathrm{Tr}\left(\sigma_i\sigma_j\right) \tag{23}$$
$$= \frac{1}{2}A_iB_j\left(2\delta_{ij}\right) \tag{24}$$
$$= A_iB_i \tag{25}$$
$$= \mathbf{A}\cdot\mathbf{B} \tag{26}$$

where we used 13 to get the fourth line. This gives us the first term on the RHS of 2.

For the other three σ_i coefficients, we can use a similar argument. Consider σ_x.

$$m_x = \frac{1}{2}\mathrm{Tr}\left(M\sigma_x\right) \tag{27}$$
$$= \frac{1}{2}A_iB_j\mathrm{Tr}\left(\sigma_i\sigma_j\sigma_x\right) \tag{28}$$

From 6 we see that $\sigma_i\sigma_j$ can always be written as a single Pauli matrix σ_α. Thus the product of 3 Pauli matrices $\sigma_i\sigma_j\sigma_x$ can be reduced to a product of 2: $\pm\sigma_\alpha\sigma_x$ (the plus or minus sign is determined by the order in which we multiply the two matrices σ_i and σ_j). However, from 13, we see that the trace of $\sigma_\alpha\sigma_x$ is non-zero only if $\alpha = x$. The only way this can happen is if either $i = y$ and $j = z$ or $i = z$ and $j = y$. Therefore we have

$$m_x = \frac{1}{2}A_yB_z\mathrm{Tr}\left(\sigma_y\sigma_z\sigma_x\right) + \frac{1}{2}A_zB_y\mathrm{Tr}\left(\sigma_z\sigma_y\sigma_x\right) \tag{29}$$

(Repeated indices are *not* summed here!) From 3 we have

$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x \tag{30}$$

Thus

$$\mathrm{Tr}\left(\sigma_y\sigma_z\sigma_x\right) = i\mathrm{Tr}\left(\sigma_x^2\right) = 2i$$

Therefore

$$m_x = i\left(A_yB_z - A_zB_y\right) \tag{31}$$

and m_x is the x component of $i\left(\mathbf{A}\times\mathbf{B}\right)$. A similar argument gives m_y and m_z, so putting everything together we again arrive at 2.

COMMENTS

Remark 1. Danyel Cavazos

Nov 12, 2017 9:24 PM

Hi!

I'd like to ask something in this page.

How do we go from eq. 22 to eq. 23? I.e., how do we know that when we evaluate Tr(A_i s_i B_j s_j) we can take A_i and B_j out of the trace operation?

Thank you so much!

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A and **B** are ordinary vectors whose components are just numbers, not matrices, so they can be taken outside the trace operation.

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Danyel Cavazos

Nov 13, 2017 5:15 PM

That's what I imagined, but then that means that we should beware of using this identity when A or B is replaced by vector operators like L or S, right?

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You might be able to prove it for the case where **A** and **B** are matrices, since any 2×2 matrix can be written as a linear combination of the Pauli matrices and the unit matrix, but it looks like it would get quite messy. I guess we can just use the first proof above which seems to work in general.

PINGBACKS

Pingback: General 2x2 matrix in terms of pauli matrices

Pingback: Projection operators for spin-1/2 + spin-1/2