

## PAULI MATRICES: A USEFUL IDENTITY

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.  
Chapter 14, Exercise 14.3.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The three components of the spin operator  $\mathbf{S}$  for spin  $\frac{1}{2}$  can be expressed in terms of the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

We can derive an identity involving the Pauli matrices:

$$(\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = (\mathbf{A} \cdot \mathbf{B})I + i(\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma} \quad (2)$$

One way of proving this is to use the commutation relations for the Pauli matrices. We have

$$[\sigma_i, \sigma_j]_+ = 2\delta_{ij}I \quad (3)$$

$$[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k \quad (4)$$

where  $\varepsilon_{ijk}$  is the Levi-Civita antisymmetric tensor.

We therefore have

$$\sigma_i \sigma_j = \frac{1}{2} \left( [\sigma_i, \sigma_j]_+ + [\sigma_i, \sigma_j] \right) \quad (5)$$

$$= \delta_{ij}I + i \sum_k \varepsilon_{ijk} \sigma_k \quad (6)$$

Using the summation convention where repeated indices are summed from 1 to 3 (that is, over  $x, y$  and  $z$ ):

$$(\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = A_i \sigma_i B_j \sigma_j \quad (7)$$

$$= A_i B_j \sigma_i \sigma_j \quad (8)$$

$$= A_i B_j (\delta_{ij} I + i \varepsilon_{ijk} \sigma_k) \quad (9)$$

$$= A_i B_j I + i \varepsilon_{ijk} A_i B_j \sigma_k \quad (10)$$

$$= (\mathbf{A} \cdot \mathbf{B}) I + i (\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma} \quad (11)$$

where the last term on the RHS follows from writing the vector cross product in terms of  $\varepsilon_{ijk}$ . [Note that in the second line, we've assumed that  $\mathbf{B}$  commutes with  $\boldsymbol{\sigma}$ .]

Another way of deriving this result is as follows. First, we add the  $2 \times 2$  identity matrix  $I$  to the set of Pauli matrices, calling it  $\sigma_0 \equiv I$ . Then, because we have four independent matrices (Shankar shows they are linearly independent in his equations 14.3.40-41) each with 4 entries, we can write any  $2 \times 2$  complex matrix as a linear combination of the  $\sigma_\alpha$  (where a Greek subscript ranges from 0 to 3). That is, for a general  $2 \times 2$  matrix  $M$

$$M = \sum_{\alpha} m_{\alpha} \sigma_{\alpha} \quad (12)$$

From the trace identities

$$\text{Tr}(\sigma_{\alpha} \sigma_{\beta}) = 2 \delta_{\alpha\beta} \quad (13)$$

we can find  $m_{\alpha}$  by right-multiplying by  $\sigma_{\beta}$  and taking the trace:

$$\text{Tr}(M \sigma_{\beta}) = \sum_{\alpha} m_{\alpha} \text{Tr}(\sigma_{\alpha} \sigma_{\beta}) \quad (14)$$

$$= 2 \sum_{\alpha} m_{\alpha} \delta_{\alpha\beta} \quad (15)$$

$$= 2 m_{\beta} \quad (16)$$

Thus

$$m_{\alpha} = \frac{1}{2} \text{Tr}(M \sigma_{\alpha}) \quad (17)$$

Returning to 2, we can identify (again using the summation convention):

$$M = (\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) \quad (18)$$

$$= A_i \sigma_i B_j \sigma_j \quad (19)$$

$$= m_{\alpha} \sigma_{\alpha} \quad (20)$$

For  $\alpha = 0$  we have

$$m_0 = \frac{1}{2} \text{Tr}(M\sigma_0) \quad (21)$$

$$= \frac{1}{2} \text{Tr}(M) \quad (22)$$

$$= \frac{1}{2} A_i B_j \text{Tr}(\sigma_i \sigma_j) \quad (23)$$

$$= \frac{1}{2} A_i B_j (2\delta_{ij}) \quad (24)$$

$$= A_i B_i \quad (25)$$

$$= \mathbf{A} \cdot \mathbf{B} \quad (26)$$

where we used 13 to get the fourth line. This gives us the first term on the RHS of 2.

For the other three  $\sigma_i$  coefficients, we can use a similar argument. Consider  $\sigma_x$ .

$$m_x = \frac{1}{2} \text{Tr}(M\sigma_x) \quad (27)$$

$$= \frac{1}{2} A_i B_j \text{Tr}(\sigma_i \sigma_j \sigma_x) \quad (28)$$

From 6 we see that  $\sigma_i \sigma_j$  can always be written as a single Pauli matrix  $\sigma_\alpha$ . Thus the product of 3 Pauli matrices  $\sigma_i \sigma_j \sigma_x$  can be reduced to a product of 2:  $\pm \sigma_\alpha \sigma_x$  (the plus or minus sign is determined by the order in which we multiply the two matrices  $\sigma_i$  and  $\sigma_j$ ). However, from 13, we see that the trace of  $\sigma_\alpha \sigma_x$  is non-zero only if  $\alpha = x$ . The only way this can happen is if either  $i = y$  and  $j = z$  or  $i = z$  and  $j = y$ . Therefore we have

$$m_x = \frac{1}{2} A_y B_z \text{Tr}(\sigma_y \sigma_z \sigma_x) + \frac{1}{2} A_z B_y \text{Tr}(\sigma_z \sigma_y \sigma_x) \quad (29)$$

(Repeated indices are *not* summed here!) From 3 we have

$$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x \quad (30)$$

Thus

$$\text{Tr}(\sigma_y \sigma_z \sigma_x) = i \text{Tr}(\sigma_x^2) = 2i$$

Therefore

$$m_x = i(A_y B_z - A_z B_y) \quad (31)$$

and  $m_x$  is the  $x$  component of  $i(\mathbf{A} \times \mathbf{B})$ . A similar argument gives  $m_y$  and  $m_z$ , so putting everything together we again arrive at 2.

## COMMENTS

*Remark 1.* Danyel Cavazos

Nov 12, 2017 9:24 PM

Hi!

I'd like to ask something in this page.

How do we go from eq. 22 to eq. 23? I.e., how do we know that when we evaluate  $\text{Tr}(A_i s_i B_j s_j)$  we can take  $A_i$  and  $B_j$  out of the trace operation?

Thank you so much!

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**A** and **B** are ordinary vectors whose components are just numbers, not matrices, so they can be taken outside the trace operation.

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Danyel Cavazos

Nov 13, 2017 5:15 PM

That's what I imagined, but then that means that we should beware of using this identity when **A** or **B** is replaced by vector operators like **L** or **S**, right?

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You might be able to prove it for the case where **A** and **B** are matrices, since any  $2 \times 2$  matrix can be written as a linear combination of the Pauli matrices and the unit matrix, but it looks like it would get quite messy. I guess we can just use the first proof above which seems to work in general.

## PINGBACKS

Pingback: General 2x2 matrix in terms of pauli matrices

Pingback: Projection operators for spin-1/2 + spin-1/2