**ROTATION OF SPINOR ABOUT ARBITRARY DIRECTION**

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.6.

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Just as orbital angular momentum operator $\mathbf{L}$ is the generator of rotations, the spin operator $\mathbf{S}$ can also be used as the generator of rotations in spin space by means of the unitary operator

$$U[R(\theta)] = e^{-i\theta\cdot\mathbf{S}/\hbar} = e^{-i\theta\cdot\sigma/2} \tag{1}$$

where we've written the operator in terms of the Pauli matrices $\sigma$, the components of which are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

For a spin pointing the direction $\hat{n}$, where $\hat{n}$ is defined in terms of the spherical angles as

$$\hat{n} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{3}$$

the corresponding eigenvectors of the operator $\hat{n}\cdot\mathbf{S}$ are

$$|\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{4}$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{5}$$

If we start with spin pointing in the $+z$ direction, then it is in the state

$$\left|s_z = \frac{\hbar}{2}\right\rangle = \frac{\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6}$$

then it should be possible to rotate this state into the general state 4 by applying the correct rotation operators in sequence.

Suppose we first rotate the state by an angle $\theta$ about the $y$ axis. This rotates the axis of spin so that it lies in the $xz$ plane in the first quadrant (that is, positive $x$ and positive $z$), making an angle $\theta$ with the $z$ axis. We can now rotate again by an angle $\phi$ about the (original) $z$ axis. The axis of spin now points in the direction given by $\hat{n}$ in 3. That is, it should be true that

$$|\hat{n}+\rangle = U[R(\phi\hat{\mathbf{z}})]\,U[R(\theta\hat{\mathbf{y}})]\begin{bmatrix}1\\0\end{bmatrix} \tag{7}$$

In order to verify this by direct calculation, we need an explicit form for $U$. This is derived by Shankar in his equation 14.3.44 so we won't repeat the derivation here. Basically, it uses the fact that $(\hat{n}\cdot\sigma)^2 = I$ and expands the exponential 1 as a power series, with the result

$$U[R(\theta)] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\left(\hat{\theta}\cdot\sigma\right) \tag{8}$$

We can use this formula to do the calculation.

$$U[R(\theta\hat{\mathbf{y}})]\begin{bmatrix}1\\0\end{bmatrix} = \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y\right]\begin{bmatrix}1\\0\end{bmatrix} \tag{9}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}\\0\end{bmatrix} - i\sin\frac{\theta}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \tag{10}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} \tag{11}$$

Applying the second rotation we get

$$U[R(\phi\hat{\mathbf{z}})]\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} = \left[\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z\right]\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} \tag{12}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\\ \sin\frac{\theta}{2}\cos\frac{\phi}{2}\end{bmatrix} - i\sin\frac{\phi}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} \tag{13}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)\\ \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)\end{bmatrix} \tag{14}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{bmatrix} \tag{15}$$

which agrees with 4.

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