ROTATION OF SPINOR ABOUT ARBITRARY DIRECTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.6.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Just as orbital angular momentum operator $\mathbf{L}$ is the generator of rotations, the spin operator $\mathbf{S}$ can also be used as the generator of rotations in spin space by means of the unitary operator

$$U[R(\theta)] = e^{-i\theta\cdot\mathbf{S}/\hbar} = e^{-i\theta\cdot\sigma/2} \tag{0.1}$$

where we've written the operator in terms of the Pauli matrices σ, the components of which are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{0.2}$$

For a spin pointing the direction $\hat{n}$, where $\hat{n}$ is defined in terms of the spherical angles as

$$\hat{n} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{0.3}$$

the corresponding eigenvectors of the operator $\hat{n}\cdot\mathbf{S}$ are

$$|\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{0.4}$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{0.5}$$

If we start with spin pointing in the $+z$ direction, then it is in the state

$$\left|s_z = \frac{\hbar}{2}\right\rangle = \frac{\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{0.6}$$

then it should be possible to rotate this state into the general state 0.4 by applying the correct rotation operators in sequence.

Suppose we first rotate the state by an angle θ about the y axis. This rotates the axis of spin so that it lies in the xz plane in the first quadrant (that is, positive x and positive z), making an angle θ with the z axis. We can now rotate again by an angle ϕ about the (original) z axis. The axis of spin now points in the direction given by $\hat{n}$ in 0.3. That is, it should be true that

$$|\hat{n}+\rangle = U\left[R\left(\phi\hat{\mathbf{z}}\right)\right]U\left[R\left(\theta\hat{\mathbf{y}}\right)\right]\left[\begin{array}{c}1\\0\end{array}\right] \tag{0.7}$$

In order to verify this by direct calculation, we need an explicit form for U. This is derived by Shankar in his equation 14.3.44 so we won't repeat the derivation here. Basically, it uses the fact that $(\hat{n}\cdot\boldsymbol{\sigma})^2 = I$ and expands the exponential 0.1 as a power series, with the result

$$U\left[R\left(\theta\right)\right] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\left(\hat{\theta}\cdot\boldsymbol{\sigma}\right) \tag{0.8}$$

We can use this formula to do the calculation.

$$U\left[R\left(\theta\hat{\mathbf{y}}\right)\right]\left[\begin{array}{c}1\\0\end{array}\right] = \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y\right]\left[\begin{array}{c}1\\0\end{array}\right] \tag{0.9}$$

$$= \left[\begin{array}{c}\cos\frac{\theta}{2}\\0\end{array}\right] - i\sin\frac{\theta}{2}\left[\begin{array}{cc}0 & -i\\i & 0\end{array}\right]\left[\begin{array}{c}1\\0\end{array}\right] \tag{0.10}$$

$$= \left[\begin{array}{c}\cos\frac{\theta}{2}\\\sin\frac{\theta}{2}\end{array}\right] \tag{0.11}$$

Applying the second rotation we get

$$U\left[R\left(\phi\hat{\mathbf{z}}\right)\right]\left[\begin{array}{c}\cos\frac{\theta}{2}\\\sin\frac{\theta}{2}\end{array}\right] = \left[\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z\right]\left[\begin{array}{c}\cos\frac{\theta}{2}\\\sin\frac{\theta}{2}\end{array}\right] \tag{0.12}$$

$$= \left[\begin{array}{c}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\\\sin\frac{\theta}{2}\cos\frac{\phi}{2}\end{array}\right] - i\sin\frac{\phi}{2}\left[\begin{array}{cc}1 & 0\\0 & -1\end{array}\right]\left[\begin{array}{c}\cos\frac{\theta}{2}\\\sin\frac{\theta}{2}\end{array}\right] \tag{0.13}$$

$$= \left[\begin{array}{c}\cos\frac{\theta}{2}\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)\\\sin\frac{\theta}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)\end{array}\right] \tag{0.14}$$

$$= \left[\begin{array}{c}\cos\frac{\theta}{2}e^{-i\phi/2}\\\sin\frac{\theta}{2}e^{i\phi/2}\end{array}\right] \tag{0.15}$$

which agrees with 0.4.

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