ROTATION OF SPINOR ABOUT ARBITRARY DIRECTION

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Just as orbital angular momentum operator $L$ is the generator of rotations, the spin operator $S$ can also be used as the generator of rotations in spin space by means of the unitary operator

$$U[R(\theta)] = e^{-i\theta S/\hbar} = e^{-i\theta \sigma/2}$$

where we’ve written the operator in terms of the Pauli matrices $\sigma$, the components of which are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For a spin pointing the direction $\hat{n}$, where $\hat{n}$ is defined in terms of the spherical angles as

$$\hat{n} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$$

the corresponding eigenvectors of the operator $\hat{n} \cdot S$ are

$$|\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

If we start with spin pointing in the $+z$ direction, then it is in the state

$$|s_z = \frac{\hbar}{2}\rangle = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then it should be possible to rotate this state into the general state by applying the correct rotation operators in sequence.

Suppose we first rotate the state by an angle $\theta$ about the $y$ axis. This rotates the axis of spin so that it lies in the $xz$ plane in the first quadrant
(that is, positive $x$ and positive $z$), making an angle $\theta$ with the $z$ axis. We can now rotate again by an angle $\phi$ about the (original) $z$ axis. The axis of spin now points in the direction given by $\hat{n}$ in $\text{Eq}$. That is, it should be true that

$$|\hat{n}+\rangle = U[R(\phi\hat{\omega})]U[R(\theta\hat{\gamma})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7}$$

In order to verify this by direct calculation, we need an explicit form for $U$. This is derived by Shankar in his equation 14.3.44 so we won’t repeat the derivation here. Basically, it uses the fact that $(\hat{n} \cdot \sigma)^2 = I$ and expands the exponential as a power series, with the result

$$U[R(\theta)] = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} (\hat{\theta} \cdot \sigma) \tag{8}$$

We can use this formula to do the calculation.

$$U[R(\theta\hat{\gamma})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \sigma_y \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{9}$$

$$\quad = \begin{bmatrix} \cos \frac{\theta}{2} \\ 0 \end{bmatrix} - i \sin \frac{\theta}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

$$\quad = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \tag{11}$$

Applying the second rotation we get

$$U[R(\phi\hat{\omega})] \begin{bmatrix} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} \cos \frac{\phi}{2} I - i \sin \frac{\phi}{2} \sigma_z \\ \sin \frac{\phi}{2} \cos \frac{\phi}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \tag{12}$$

$$\quad = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \\ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \tag{13}$$

$$\quad = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \tag{14}$$

which agrees with $\text{Eq}$.4.