

ROTATION OF SPINOR ABOUT ARBITRARY DIRECTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.6.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Just as orbital angular momentum operator \mathbf{L} is the generator of rotations, the spin operator \mathbf{S} can also be used as the generator of rotations in spin space by means of the unitary operator

$$U[R(\theta)] = e^{-i\theta \cdot \mathbf{S}/\hbar} = e^{-i\theta \cdot \boldsymbol{\sigma}/2} \quad (1)$$

where we've written the operator in terms of the Pauli matrices $\boldsymbol{\sigma}$, the components of which are

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (2)$$

For a spin pointing the direction \hat{n} , where \hat{n} is defined in terms of the spherical angles as

$$\hat{n} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \quad (3)$$

the corresponding eigenvectors of the operator $\hat{n} \cdot \mathbf{S}$ are

$$|\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (4)$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (5)$$

If we start with spin pointing in the $+z$ direction, then it is in the state

$$\left| s_z = \frac{\hbar}{2} \right\rangle = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (6)$$

then it should be possible to rotate this state into the general state 4 by applying the correct rotation operators in sequence.

Suppose we first rotate the state by an angle θ about the y axis. This rotates the axis of spin so that it lies in the xz plane in the first quadrant (that

is, positive x and positive z), making an angle θ with the z axis. We can now rotate again by an angle ϕ about the (original) z axis. The axis of spin now points in the direction given by \hat{n} in 3. That is, it should be true that

$$|\hat{n}+\rangle = U[R(\phi\hat{z})]U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (7)$$

In order to verify this by direct calculation, we need an explicit form for U . This is derived by Shankar in his equation 14.3.44 so we won't repeat the derivation here. Basically, it uses the fact that $(\hat{n} \cdot \sigma)^2 = I$ and expands the exponential 1 as a power series, with the result

$$U[R(\theta)] = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} (\hat{\theta} \cdot \sigma) \quad (8)$$

We can use this formula to do the calculation.

$$U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left[\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \sigma_y \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (9)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} \\ 0 \end{bmatrix} - i \sin \frac{\theta}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (10)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \quad (11)$$

Applying the second rotation we get

$$U[R(\phi\hat{z})] \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} = \left[\cos \frac{\phi}{2} I - i \sin \frac{\phi}{2} \sigma_z \right] \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \quad (12)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} \cos \frac{\phi}{2} \\ \sin \frac{\theta}{2} \cos \frac{\phi}{2} \end{bmatrix} - i \sin \frac{\phi}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \quad (13)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} \left(\cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right) \\ \sin \frac{\theta}{2} \left(\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right) \end{bmatrix} \quad (14)$$

$$= \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (15)$$

which agrees with 4.

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