

ROTATION OF SPINOR ABOUT ARBITRARY DIRECTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.6.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Just as orbital angular momentum operator \mathbf{L} is the generator of rotations, the spin operator \mathbf{S} can also be used as the generator of rotations in spin space by means of the unitary operator

$$(1) \quad U[R(\boldsymbol{\theta})] = e^{-i\boldsymbol{\theta}\cdot\mathbf{S}/\hbar} = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}$$

where we've written the operator in terms of the Pauli matrices $\boldsymbol{\sigma}$, the components of which are

$$(2) \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For a spin pointing the direction \hat{n} , where \hat{n} is defined in terms of the spherical angles as

$$(3) \quad \hat{n} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$$

the corresponding eigenvectors of the operator $\hat{n} \cdot \mathbf{S}$ are

$$(4) \quad |\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix}$$

$$(5) \quad |\hat{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix}$$

If we start with spin pointing in the $+z$ direction, then it is in the state

$$(6) \quad \left|s_z = \frac{\hbar}{2}\right\rangle = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then it should be possible to rotate this state into the general state 4 by applying the correct rotation operators in sequence.

Suppose we first rotate the state by an angle θ about the y axis. This rotates the axis of spin so that it lies in the xz plane in the first quadrant (that is, positive x and positive z), making an angle θ with the z axis. We can now rotate again by an angle ϕ about the (original) z axis. The axis of spin now points in the direction given by \hat{n} in 3. That is, it should be true that

$$(7) \quad |\hat{n}+\rangle = U[R(\phi\hat{z})]U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In order to verify this by direct calculation, we need an explicit form for U . This is derived by Shankar in his equation 14.3.44 so we won't repeat the derivation here. Basically, it uses the fact that $(\hat{n} \cdot \sigma)^2 = I$ and expands the exponential 1 as a power series, with the result

$$(8) \quad U[R(\theta)] = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} (\hat{\theta} \cdot \sigma)$$

We can use this formula to do the calculation.

$$(9) \quad U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left[\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \sigma_y \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(10) \quad = \begin{bmatrix} \cos \frac{\theta}{2} \\ 0 \end{bmatrix} - i \sin \frac{\theta}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(11) \quad = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix}$$

Applying the second rotation we get

$$(12)$$

$$U[R(\phi\hat{z})] \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} = \left[\cos \frac{\phi}{2} I - i \sin \frac{\phi}{2} \sigma_z \right] \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix}$$

$$(13) \quad = \begin{bmatrix} \cos \frac{\theta}{2} \cos \frac{\phi}{2} \\ \sin \frac{\theta}{2} \cos \frac{\phi}{2} \end{bmatrix} - i \sin \frac{\phi}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix}$$

$$(14) \quad = \begin{bmatrix} \cos \frac{\theta}{2} \left(\cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right) \\ \sin \frac{\theta}{2} \left(\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right) \end{bmatrix}$$

$$(15) \quad = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

which agrees with 4.

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