PAULI MATRICES: EXAMPLES OF LINEAR COMBINATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.7.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Here are a few examples of calculations using the Pauli matrices σ, the components of which are

$$\sigma_x=\left[\begin{array}{cc}0 & 1\\1 & 0\end{array}\right];\quad\sigma_y=\left[\begin{array}{cc}0 & -i\\i & 0\end{array}\right];\quad\sigma_z=\left[\begin{array}{cc}1 & 0\\0 & -1\end{array}\right]\tag{1}$$

From Shankar's equation 14.3.44, we know that the unitary rotation operator can be written as

$$U\left[R\left(\theta\right)\right]=e^{-i\theta\cdot\sigma/2}\tag{2}$$

$$=\cos\frac{\theta}{2}I-i\sin\frac{\theta}{2}\left(\hat{\theta}\cdot\sigma\right)\tag{3}$$

Example 1. Find $\left(I+i\sigma_x\right)^{1/2}$. As usual, the square root of a matrix M is the matrix $M^{1/2}$ such that $M^{1/2}M^{1/2}=M$. To solve this, we would like to express $I+i\sigma_x$ in the form 2, from which we can find the square root by simply dividing the exponent by 2. We first express it in the form 3, from which we see that we need an angle θ such that

$$\cos\frac{\theta}{2}=-\sin\frac{\theta}{2}\tag{4}$$

This is valid if

$$\theta = \frac{3\pi}{2}\tag{5}$$

$$\cos\frac{\theta}{2} = -\frac{\sqrt{2}}{2}=-\sin\frac{\theta}{2}\tag{6}$$

This gives

$$U = -\frac{\sqrt{2}}{2}(I+i\sigma_x) \tag{7}$$

$$I+i\sigma_x = -\sqrt{2}e^{i\sigma_x 3\pi/4} \tag{8}$$

$$= \sqrt{2}e^{i\pi}e^{i\sigma_x 3\pi/4} \tag{9}$$

Therefore

$$(I+i\sigma_x)^{1/2} = 2^{1/4}e^{i\pi/2}e^{i\sigma_x 3\pi/8} \tag{10}$$

$$= 2^{1/4}i\left(\cos\frac{3\pi}{8}I - i\sin\frac{3\pi}{8}\sigma_x\right) \tag{11}$$

We can check this by evaluating the cos and sin using the half-angle formulas

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \tag{12}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \tag{13}$$

We therefore have

$$\sin\frac{3\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}} \tag{14}$$

$$\cos\frac{3\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}} \tag{15}$$

Plugging these into 11 we have

$$(I+i\sigma_x)^{1/2} = \frac{1}{2^{3/4}}\left[\begin{array}{cc} i\sqrt{2-\sqrt{2}} & \sqrt{2+\sqrt{2}} \\ \sqrt{2+\sqrt{2}} & i\sqrt{2-\sqrt{2}} \end{array}\right] \tag{16}$$

Squaring this gives

$$I+i\sigma_x=\frac{1}{2^{3/2}}\left[\begin{array}{cc}-\left(2-\sqrt{2}\right)+2+\sqrt{2} & 2i\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}}\\ 2i\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}} & 2+\sqrt{2}-\left(2-\sqrt{2}\right)\end{array}\right] \tag{17}$$

$$=\frac{1}{2\sqrt{2}}\left[\begin{array}{cc}2\sqrt{2} & 2\sqrt{2}i\\ 2\sqrt{2}i & 2\sqrt{2}\end{array}\right] \tag{18}$$

$$=\left[\begin{array}{cc}1 & i\\ i & 1\end{array}\right] \tag{19}$$

which is correct.

[Incidentally, 11 is different from Shankar's answer in the back of the book, but both are correct as can be verified by squaring Shankar's answer. Unlike ordinary complex numbers, a 2×2 matrix can have more than 2 square roots.]

Example 2. Find $(2I+\sigma_x)^{-1}$. In principle, we could solve this the same way as in Example 1, but this time we would need to find θ such that $\cos\frac{\theta}{2}=-2\sin\frac{\theta}{2}$. This doesn't give a 'nice' value of θ (that is, a value that is some nice multiple of π). It seems easier to just calculate the matrix and then take its inverse using the standard formula for the inverse of a 2×2 matrix. We can then convert this back to a linear combination of Pauli matrices using the formula for a matrix M:

$$M=\left[\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right] \tag{20}$$

$$=\frac{1}{2}\left[(\alpha+\delta)I+(\beta+\gamma)\sigma_x+i(\beta-\gamma)\sigma_y+(\alpha-\delta)\sigma_z\right] \tag{21}$$

We get

$$2I+\sigma_x=\left[\begin{array}{cc}2 & 1\\ 1 & 2\end{array}\right] \tag{22}$$

The inverse of a matrix is given by

$$\left[\begin{array}{cc}a & b\\ c & d\end{array}\right]^{-1}=\frac{1}{ad-bc}\left[\begin{array}{cc}d & -b\\ -c & a\end{array}\right] \tag{23}$$

so

$$(2I+\sigma_x)^{-1}=\left[\begin{array}{cc}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{array}\right] \tag{24}$$

Using 21 we find

$$(2I+\sigma_x)^{-1}=\frac{1}{6}(4I-2\sigma_x)=\frac{1}{3}(2I-\sigma_x) \tag{25}$$

We can check this by multiplication

$$\frac{1}{3}(2I-\sigma_x)(2I+\sigma_x) = \frac{1}{3}\left(4I-\sigma_x^2\right) \tag{26}$$

$$= \frac{1}{3}(4I-I) \tag{27}$$

$$= I \tag{28}$$

where we used $\sigma_x^2=I$ to get the second line.

Example 3. Find σ_x^{-1}. Since $\sigma_x^2=I$, $\sigma_x^{-1}=\sigma_x$.