

PAULI MATRICES: COMMUTATION AND ANTICOMMUTATION PROPERTIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.8.

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Here are a couple of theorems concerning the Pauli matrices σ , the components of which are

$$(0.1) \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Both theorems arise from the fact that an arbitrary 2×2 matrix can be written as a linear combination of the Pauli matrices and the unit matrix:

$$(0.2) \quad M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$(0.3) \quad = \frac{1}{2} [(\alpha + \delta)I + (\beta + \gamma)\sigma_x + i(\beta - \gamma)\sigma_y + (\alpha - \delta)\sigma_z]$$

We'll also need the commutation and anticommutation relations

$$(0.4) \quad [\sigma_i, \sigma_j]_+ = 2\delta_{ij}I$$

$$(0.5) \quad [\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k$$

Theorem 1. *Any matrix that commutes with σ (that is, it commutes with all 3 components of σ) is a multiple of the unit matrix.*

Proof. First, since I commutes with every matrix, it commutes with σ . Now, from 0.5, any one of the Pauli matrices does *not* commute with the other two Pauli matrices, so M cannot have any component that is one of the Pauli matrices. From 0.3, this means that

$$(0.6) \quad \beta + \gamma = 0$$

$$(0.7) \quad \beta - \gamma = 0$$

$$(0.8) \quad \alpha - \delta = 0$$

The first two conditions say that $\beta = \gamma = -\gamma$ which implies $\beta = \gamma = 0$ and the last condition gives us $\alpha = \delta$, so M must be a multiple of the unit matrix. \square

Theorem 2. *There is no matrix (apart from the zero matrix) that anticommutes with all 3 Pauli matrices.*

Proof. Since I doesn't anticommute with any matrix, M cannot contain a component with I . From 0.4, the anticommutator of two Pauli matrices is zero only if the two matrices are different. Therefore, if M contains a non-zero component for any one, say σ_x , of the Pauli matrices then M will not anticommute with σ_x . The same argument applies to the other two Pauli matrices, so there is no M that anticommutes with all 3 Pauli matrices. \square