PAULI MATRICES: COMMUTATION AND ANTICOMMUTATION PROPERTIES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.8.

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Here are a couple of theorems concerning the Pauli matricesσ, the components of which are

$$\sigma_x=\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right];\quad\sigma_y=\left[\begin{array}{cc}0 & -i\\ i & 0\end{array}\right];\quad\sigma_z=\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right]\tag{1}$$

Both theorems arise from the fact that an arbitrary 2×2 matrix can be written as a linear combination of the Pauli matrices and the unit matrix:

$$M = \left[\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right]\tag{2}$$

$$= \frac{1}{2}\left[\left(\alpha+\delta\right)I+\left(\beta+\gamma\right)\sigma_x+i\left(\beta-\gamma\right)\sigma_y+\left(\alpha-\delta\right)\sigma_z\right]\tag{3}$$

We'll also need the commutation and anticommutationsrelations

$$\left[\sigma_i,\sigma_j\right]_+ = 2\delta_{ij}I\tag{4}$$

$$\left[\sigma_i,\sigma_j\right] = 2i\sum_k\varepsilon_{ijk}\sigma_k\tag{5}$$

Theorem 1. *Any matrix that commutes with σ (that is, it commutes with all 3 components of σ) is a multiple of the unit matrix.*

Proof. First, since I commutes with every matrix, it commutes with σ. Now, from 5, any one of the Pauli matrices does *not* commute with the other two Pauli matrices, so M cannot have any component that is one of the Pauli matrices. From 3, this means that

$$\beta+\gamma = 0 \quad (6)$$

$$\beta-\gamma = 0 \quad (7)$$

$$\alpha-\delta = 0 \quad (8)$$

The first two conditions say that $\beta=\gamma=-\gamma$ which implies $\beta=\gamma=0$ and the last condition gives us $\alpha=\delta$, so M must be a multiple of the unit matrix. □

Theorem 2. *There is no matrix (apart from the zero matrix) that anticommutes with all 3 Pauli matrices.*

Proof. Since I doesn't anticommute with any matrix, M cannot contain a component with I. From 4, the anticommutator of two Pauli matrices is zero only if the two matrices are different. Therefore, if M contains a non-zero component for any one, say σ_x, of the Pauli matrices then M will not anticommute with σ_x. The same argument applies to the other two Pauli matrices, so there is no M that anticommutes with all 3 Pauli matrices. □

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