AVERAGE RATE OF CHANGE OF ANGULAR MOMENTUM IN MAGNETIC FIELD

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.1.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In classical electrodynamics, the torque on a magnetic moment μ in a constant magnetic field $\mathbf{B}$ is given by (using Shankar's notation):

$$\mathbf{T} = \mu \times \mathbf{B} \tag{1}$$

We can relate the magnetic moment to the angular momentum of a (classically) spinning charged object by introducing the gyromagnetic ratio

$$\gamma \equiv \frac{\mu}{l} \tag{2}$$

If we apply this to a single particle of charge q and mass m travelling at constant speed v around a circular orbit, then its angular momentum is

$$l = mvr \tag{3}$$

The magnetic moment can be calculated by taking the charge q to be smeared out over the circumference of the circle, giving a linear charge density of

$$\lambda = \frac{q}{2\pi r} \tag{4}$$

Since the loop is spinning with speed v, the current (rate at which charge passed a fixed point on the circle) is

$$I = \lambda v = \frac{q}{2\pi r} r\omega = \frac{q}{2\pi}\omega \tag{5}$$

where

$$\omega = \frac{2\pi}{P} = \frac{2\pi v}{2\pi r} = \frac{v}{r} \tag{6}$$

is the angular frequency (P is the period, or time it takes for one complete orbit).

The magnetic moment is defined as

$$\mu \equiv \frac{I}{c}\mathbf{a} \tag{7}$$

where **a** is the area of the loop, whose direction is determined by using the right-hand rule on the direction of the current around the loop. Thus if the current is travelling counterclockwise when viewed from above, **a** points upwards. (The speed of light c enters because Shankar is using CGS units.) The magnetic moment here is then

$$\begin{aligned}
\mu &= \frac{qv}{2\pi r}\frac{\pi r^2}{c}\hat{\mathbf{a}} &(8)\\
&= \left(\frac{q}{2mc}\right)(mvr\hat{\mathbf{a}}) &(9)\\
&= \left(\frac{q}{2mc}\right)\mathbf{l} &(10)
\end{aligned}$$

where **l** is the angular momentum vector. In this case, the gyromagnetic ratio is

$$\gamma = \frac{q}{2mc} \tag{11}$$

In this case, the torque 1 is given by

$$\mathbf{T} = \gamma\mathbf{l}\times\mathbf{B} \tag{12}$$

The interaction energy (between the angular momentum and magnetic field) is given by

$$H_{int} = \int T(\theta)\,d\theta \tag{13}$$

where the torque is given as a function of the angle between μ and **B** in 1, so that

$$T = \mu B\sin\theta \tag{14}$$

Doing the integral (neglecting the constant of integration) we have

$$H_{int} = -\mu B\cos\theta = -\mu\cdot\mathbf{B} \tag{15}$$

H_{int} is minimized when μ and **B** are parallel, so the torque's effect is to try to bring these two vectors into alignment. This assumes that the magnetic moment doesn't actually involve any angular momentum, which obviously isn't the case with our rotating loop example above. In that case, the torque causes a precession about the direction of **B**, which we can see as follows.

The angular version of Newton's law, relating torque and angular momentum, is

$$\mathbf{T}=\frac{d\mathbf{l}}{dt}=\gamma\mathbf{l}\times\mathbf{B} \tag{16}$$

Since the cross product is perpendicular to both its constituent vectors, the change in $\mathbf{l}$ is always perpendicular to $\mathbf{l}$ itself. The effect can be seen by looking at Shankar's Figure 14.2 (too much effort to reproduce that here), in which we can see that

$$\Delta\mathbf{l} = \gamma(\mathbf{l}\times\mathbf{B})\Delta t \tag{17}$$

$$\Delta l = \gamma lB\sin\theta\Delta t \tag{18}$$

where θ is the angle between $\mathbf{l}$ and $\mathbf{B}$, and $\Delta\mathbf{l}$ is tangent to the circle of radius $l\sin\theta$ that lies in the plane perpendicular to $\mathbf{B}$. The net effect is that $\mathbf{l}$ precesses about the direction of $\mathbf{B}$, so that the magnitude of angular momentum remains constant, but its direction changes at a constant rate. The change in azimuthal angle $\Delta\phi$ in time Δt is

$$\Delta\phi=\frac{-\Delta l}{l\sin\theta}=-\gamma B\Delta t \tag{19}$$

where the minus sign is because the angular momentum precesses clockwise (as seen from above) around $\mathbf{B}$. The angular frequency of precession is therefore

$$\omega_0=\frac{\Delta\phi}{\Delta t}=-\gamma B \tag{20}$$

If we include the direction of the axis of precession, which is parallel to $\mathbf{B}$, then

$$\omega_0=-\gamma\mathbf{B} \tag{21}$$

We can see that these results transfer over to quantum mechanics if we use Ehrenfest's theorem. For an interaction hamiltonian 15, we can write it as

$$H=-\gamma\mathbf{L}\cdot\mathbf{B} \tag{22}$$

We want to find the average of the angular momentum over time, so we use Ehrenfest's theorem to write

$$\frac{d\langle\mathbf{L}\rangle}{dt}=-\frac{i}{\hbar}\langle[\mathbf{L},H]\rangle \tag{23}$$

We can work out the RHS using the commutators of angular momentum:

$$\left[L_i, L_j\right] = i\hbar \sum_k \varepsilon_{ijk} L_k \tag{24}$$

As we're dealing with a vector operator, we can work out each component separately. For L_x we get, assuming that **B** is independent of position (and thus commutes with **L**):

$$-\frac{i}{\hbar}\left[L_x, H\right] = \frac{i\gamma}{\hbar}\left[L_x, L_xB_x + L_yB_y + L_zB_z\right] \tag{25}$$

$$= \frac{i\gamma}{\hbar}\left(\left[L_x, L_x\right]B_x + \left[L_x, L_y\right]B_y + \left[L_x, L_z\right]B_z\right) \tag{26}$$

$$= -\gamma\left(0 + L_zB_y - L_yB_z\right) \tag{27}$$

$$= \gamma\left(\mathbf{L}\times\mathbf{B}\right)_x \tag{28}$$

$$= \left(\mu\times\mathbf{B}\right)_x \tag{29}$$

The other two components work out similarly, so we have

$$-\frac{i}{\hbar}\left\langle\left[\mathbf{L}, H\right]\right\rangle = \mu\times\mathbf{B} \tag{30}$$

As **B** doesn't depend on position, when we take the average over space we get

$$\frac{d\left\langle\mathbf{L}\right\rangle}{dt} = \left\langle\mu\right\rangle\times\mathbf{B} \tag{31}$$

Thus the mean of the quantum angular momentum also precessed about **B**. Since the only assumption we made was that **B** was independent of position, and all that was used in the derivation was the commutation relations of angular momentum, the result is also valid for spin angular momentum, and time-varying magnetic fields, provided they are constant over all space.

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