MAGNETIC MOMENT IN OSCILLATING MAGNETIC FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In classical electromagnetism, a magnetic moment precessesif placed in a constant magnetic field whose direction is not parallel to that of the magnetic moment. For a magnetic moment μ in a constant field $\mathbf{B}_0$, the precession has a frequency of

$$\omega_0 = -\gamma \mathbf{B}_0 \tag{1}$$

where γ is the gyromagnetic ratio.

Now suppose we view this precession in a frame of reference that is rotating about the same axis as ω_0, but with a frequency ω that may not be the same as ω_0. The precession frequency will now appear to be

$$\omega_r = \omega_0 - \omega \tag{2}$$

[Although this is a vector equation, all vectors in it have the same direction.] Comparing this with 1, we see that, in the rotating frame, the effective magnetic field is

$$\mathbf{B}_r = -\frac{1}{\gamma}\omega_r = \mathbf{B}_0 + \frac{\omega}{\gamma} \tag{3}$$

Now suppose the magnetic field is taken to be constant in the z direction with component $B_0\hat{\mathbf{z}}$, but with a small oscillating component in the xy plane, so that the total field is

$$\mathbf{B} = B\cos\omega t\hat{\mathbf{x}} - B\sin\omega t\hat{\mathbf{y}} + B_0\hat{\mathbf{z}} \tag{4}$$

where $B \ll B_0$.

This is a magnetic field that precesses about the z axis, so it's similar to the case we treated earlier, although in the earlier post we were concerned only with the behaviour of an electron in such a field, so we were interested in the quantum mechanics. The present treatment is purely classical.

If we place a magnetic moment in this field so that at $t = 0$ it's pointing in the $+z$ direction, we want to find how the magnetic moment varies with time. To analyze the problem, it's easiest to transform to a rotating frame with frequency $\boldsymbol{\omega} = -\omega\hat{\mathbf{z}}$ (minus, because it's precessing in a clockwise direction). Since the frame is rotating at the same rate as the magnetic field, the field appears frozen in this rotating frame. For simplicity, we'll assume that the field's horizontal component lies along the $+x$ direction, so the field lies in the xz plane. The z component of the field is thus effectively reduced to

$$B_z = B_0 - \frac{\omega}{\gamma} \tag{5}$$

In this frame, we therefore have a constant magnetic field given by

$$\mathbf{B}_r = B\hat{\mathbf{x}} + \left(B_0 - \frac{\omega}{\gamma}\right)\hat{\mathbf{z}} \tag{6}$$

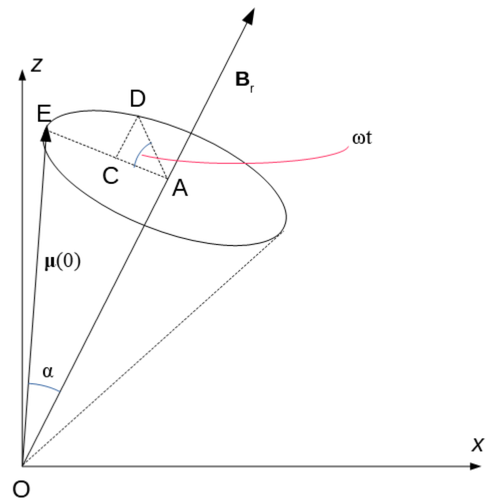
The magnetic moment should then precess about $\mathbf{B}_r$. To get the frequency ω_r of this precession, we get the magnitude of the magnetic field:

$$B_r = \sqrt{B^2 + \left(B_0 - \frac{\omega}{\gamma}\right)^2} \tag{7}$$

The precession frequency is then

$$\omega_r = -\gamma\mathbf{B}_r \tag{8}$$

Refer to the following figure (similar to Shankar's Fig. 14.3, but with a few added points) for what follows.



In the figure $\mu(0)$ is given by the vector OE, so it starts off pointing in the $+z$ direction. [Just as in Shankar's figure, we've drawn this vector so it's not quite parallel to the z axis, although in the problem $\mu(0)$ does actually point directly along the z axis. Drawing it this way makes the figure a bit easier to follow.] To get the z component of μ as it precesses about $\mathbf{B}_r$, suppose we look at μ at time t, when it has precessed through an angle ωt, so μ now lies along the vector OD (I haven't drawn the vector in the diagram since it would get too cluttered, but you can imagine the vector.) To get the z component of this vector, we look at its components parallel and perpendicular to the plane followed by the tip of μ as it precesses. This is the plane occupied by the circle in the diagram (well, ok, in the diagram it's an ellipse because we're looking at the circle from an angle). If the angle between μ and $\mathbf{B}_r$ is α, then the components of $\mu(\omega t)$ are

$$AD = \mu\sin\alpha \tag{9}$$
$$OA = \mu\cos\alpha \tag{10}$$

Note that the magnitude of μ is constant; only its direction changes by precession. The angle α between μ and $\mathbf{B}_r$ is also constant.

To get the projections of these two segments onto the z axis, we look first at the projection of OA since OA always lies in the xz plane. From the diagram

$$OA_z = (\mu\cos\alpha)\cos\alpha = \mu\cos^2\alpha \tag{11}$$

To get the z projection of AD, we first project it onto the xz plane by projecting it onto AE, giving the segment AC:

$$
\begin{aligned}
AC &= AD\cos\omega t && (12)\\
&= \mu\sin\alpha\cos\omega t && (13)
\end{aligned}
$$

We then project AC onto the z axis. The line AC makes an angle α with the x axis, so the projection introduces another factor of $\sin\alpha$:

$$AC_z = AC\sin\alpha = \mu\sin^2\alpha\cos\omega t \tag{14}$$

The z component of μ is therefore the sum of 11 and 14:

$$\mu_z = \mu\cos^2\alpha + \mu\sin^2\alpha\cos\omega t \tag{15}$$

To get the final form, we need to eliminate α which we can do from 6, since α is the angle between $\mathbf{B}_r$ and the z axis. Therefore

$$
\begin{aligned}
\sin\alpha &= \frac{B}{B_r} && (16)\\
&= \frac{B}{\sqrt{B^2+\left(B_0-\frac{\omega}{\gamma}\right)^2}} && (17)\\
&= \frac{\gamma B}{\sqrt{\gamma^2B^2+(\gamma B_0-\omega)^2}} && (18)\\
\cos\alpha &= \frac{B_0-\frac{\omega}{\gamma}}{\sqrt{B^2+\left(B_0-\frac{\omega}{\gamma}\right)^2}} && (19)\\
&= \frac{\gamma B_0-\omega}{\sqrt{\gamma^2B^2+(\gamma B_0-\omega)^2}} && (20)
\end{aligned}
$$

We can write this in terms of the frequency ω_0 by which the magnetic moment would precess if the field were constant, which is

$$\omega_0 = |\omega_0| = \gamma B_0 \tag{21}$$

So we get

$$\sin\alpha = \frac{\gamma B}{\sqrt{\gamma^2 B^2 + (\omega_0 - \omega)^2}} \tag{22}$$

$$\cos\alpha = \frac{\omega_0 - \omega}{\sqrt{\gamma^2 B^2 + (\omega_0 - \omega)^2}} \tag{23}$$

Plugging this into 15 we get

$$\mu_z(t) = \mu_z(0)\left[\frac{(\omega_0 - \omega)^2}{\gamma^2 B^2 + (\omega_0 - \omega)^2} + \frac{\gamma^2 B^2 \cos\omega t}{\gamma^2 B^2 + (\omega_0 - \omega)^2}\right] \tag{24}$$

PINGBACKS

Pingback: Spinor in oscillating magnetic field - part 1
Pingback: Spinor in oscillating magnetic field - part 2
Pingback: Effective magnetic field in rotating frame - axis of rotation not parallel