SPINOR IN OSCILLATING MAGNETIC FIELD - PART 1

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.3, Part 1.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Classically, if a magnetic moment μ is placed in a magnetic field that precesses about the z axis, the magnetic moment itself precesses. If the field is given as

$$\mathbf{B}=B\cos\omega t\hat{\mathbf{x}}-B\sin\omega t\hat{\mathbf{y}}+B_0\hat{\mathbf{z}} \tag{1}$$

then in a frame that rotates with the same frequency as the field, the magnetic field appears to be constant with value

$$\mathbf{B}_r=B\hat{\mathbf{x}}_r+\left(B_0-\frac{\omega}{\gamma}\right)\hat{\mathbf{z}} \tag{2}$$

where

$$\hat{\mathbf{x}}_r=\cos\omega t\hat{\mathbf{x}}-\sin\omega t\hat{\mathbf{y}} \tag{3}$$

is a unit vector along the x axis in the rotating frame. We now want to see how this result transfers into quantum mechanics.

We begin with the Schrödinger equation for the state $|\psi(t)\rangle$ in the lab (non-rotating) frame, which is, as usual

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle=H|\psi(t)\rangle \tag{4}$$

We'll study the case where $|\psi(t)\rangle$ is a spin $\frac{1}{2}$ state, for which the Hamiltonian is

$$H=-\gamma\mathbf{S}\cdot\mathbf{B} \tag{5}$$

We can analyze the situation in the rotating frame by applying a unitary rotation operator to the lab state. That is

$$\begin{aligned}\left|\psi_{r}(t)\right\rangle &= e^{-i\omega tS_{z}/\hbar}\left|\psi(t)\right\rangle = e^{-i\omega t\sigma_{z}/2}\left|\psi(t)\right\rangle &(6)\\ &= \left[\cos\frac{\omega t}{2}I - i\sin\frac{\omega t}{2}\sigma_{z}\right]\left|\psi(t)\right\rangle &(7)\end{aligned}$$

[It seems to me that this unitary operator is for a rotation by an angle ωt, and since the rotation of the field in 1 is given by a frequency $-\omega\hat{\mathbf{z}}$, we should really be using the rotation operator $e^{i\omega t\sigma_{z}/2}$. However if we do this (I tried) we get the wrong answer, so presumably the transformation 6 is correct.]

Our first goal is to find the Schrödinger equation for $\left|\psi_{r}(t)\right\rangle$, which involves finding the corresponding Hamiltonian. The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\left|\psi_{r}(t)\right\rangle = H_{r}\left|\psi_{r}(t)\right\rangle \tag{8}$$

Inserting 6 into the LHS and differentiating, we get

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\left|\psi_{r}(t)\right\rangle &= \frac{\hbar\omega\sigma_{z}}{2}e^{-i\omega t\sigma_{z}/2}\left|\psi(t)\right\rangle + i\hbar e^{-i\omega t\sigma_{z}/2}\frac{\partial}{\partial t}\left|\psi(t)\right\rangle &(9)\\ &= \frac{\hbar\omega\sigma_{z}}{2}\left|\psi_{r}(t)\right\rangle + e^{-i\omega t\sigma_{z}/2}H\left|\psi(t)\right\rangle &(10)\\ &= \frac{\hbar\omega\sigma_{z}}{2}\left|\psi_{r}(t)\right\rangle - e^{-i\omega t\sigma_{z}/2}\gamma\mathbf{S}\cdot\mathbf{B}\left|\psi(t)\right\rangle &(11)\end{aligned}$$

We would like the RHS to be in the form of the RHS of 8, but in the second term, the problem is that $e^{-i\omega t\sigma_{z}/2}$ does not commute with $\mathbf{S}$ so we can't just swap the $e^{-i\omega t\sigma_{z}/2}$ and $\mathbf{S}\cdot\mathbf{B}$ factors. We need to multiply out the terms and see what simplifications we can do.

In what follows, it's easier to work with the Pauli matrices defined by

$$\mathbf{S} = \frac{\hbar}{2}\sigma \tag{12}$$

We'll also need a few theorems involving σ_{i}

$$\begin{aligned}\sigma_{i}\sigma_{j} &= -\sigma_{j}\sigma_{i} &(13)\\ \sigma_{i}\sigma_{j} &= \delta_{ij}I + i\sum_{k}\varepsilon_{ijk}\sigma_{k} &(14)\end{aligned}$$

We'll also define some shorthand for the trig functions:

$$c \equiv \cos\frac{\omega t}{2} \tag{15}$$

$$s \equiv \sin\frac{\omega t}{2} \tag{16}$$

$$c_1 \equiv \cos\omega t \tag{17}$$

$$s_1 \equiv \sin\omega t \tag{18}$$

The standard double-angle formulas are

$$c_1 = c^2 - s^2 \tag{19}$$

$$s_1 = 2sc \tag{20}$$

Using 1 and 7 we have

$$-e^{-i\omega t\sigma_z/2}\gamma\mathbf{S}\cdot\mathbf{B} = -\frac{\gamma\hbar}{2}\left[B\left(c-is\sigma_z\right)\left(\sigma_x c_1-\sigma_y s_1\right)+B_0\left(c-is\sigma_z\right)\sigma_z\right] \tag{21}$$

The last term on the RHS is in the correct form since there are no commutation problems here. So we need to work on the first term, which we'll isolate here:

$$\left(c-is\sigma_z\right)\left(\sigma_x c_1-\sigma_y s_1\right)=c_1 c\sigma_x+ic_1 s\sigma_x\sigma_z-s_1 c\sigma_y-is_1 s\sigma_y\sigma_z \tag{22}$$

We can now use the identities 13 and 14 and the trig identities above to get

$$\begin{aligned}
\left(c-is\sigma_z\right)\left(\sigma_x c_1-\sigma_y s_1\right) &= \left(c^2-s^2\right)c\sigma_x+i\left(c^2-s^2\right)s\sigma_x\sigma_z-2sc^2\sigma_y-2is^2c\sigma_y\sigma_z && (23)\\
&= \left(c^2-s^2\right)c\sigma_x+i\left(c^2-s^2\right)s\sigma_x\sigma_z-2isc^2\sigma_x\sigma_z+2s^2c\sigma_x && (24)\\
&= \left(c^3-s^2c+2s^2c\right)\sigma_x+i\left(-s^3+c^2s-2sc^2\right)\sigma_x\sigma_z && (25)\\
&= \left(c^2+s^2\right)c\sigma_x-i\left(c^2+s^2\right)s\sigma_x\sigma_z && (26)\\
&= \sigma_x\left(c-is\sigma_z\right) && (27)\\
&= \sigma_x e^{-i\omega t\sigma_z/2} && (28)
\end{aligned}$$

Plugging this into 21 and then back into 11 we get

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\psi_r(t)\right\rangle &= \frac{\hbar\omega\sigma_z}{2}\left|\psi_r(t)\right\rangle - \frac{\gamma\hbar}{2}\left[B\sigma_x + B_0\sigma_z\right]e^{-i\omega t\sigma_z/2}\left|\psi(t)\right\rangle \qquad (29)\\
&= \left[\left(\omega-\gamma B_0\right)S_z - \gamma BS_x\right]\left|\psi_r(t)\right\rangle \qquad (30)
\end{aligned}
$$

Comparing this with 2, we see that we can write the result as

$$
i\hbar\frac{\partial}{\partial t}\left|\psi_r(t)\right\rangle = -\gamma\mathbf{S}\cdot\mathbf{B}_r\left|\psi_r(t)\right\rangle \qquad (31)
$$

Thus in the rotating frame, the Schrödinger equation has the same form as the classical relation, with a time-independent magnetic field $\mathbf{B}_r$.

COMMENTS

From: Petra Axolotl, 3 Jul 2018, 00:37.

The transformation 6 is indeed correct, for the following reason. - At time t, the rotating frame has rotated by $-\omega t$ relative to the rest frame. - Therefore everything in the rest frame, including the wave function $\psi(t)$, should be rotated by $+\omega t$ to get $\psi_r(t)$. - The unitary operator for a rotation by $+\omega t$ is $\exp(-i\omega tS_z/h)$.

Conclusion: $\psi_r(t) = \exp(-i\omega tS_z/h)\psi(t)$.

This is in fact the same as in certain derivations regarding translational invariance, where moving the frame by $-\Delta$ means moving the wave function by $+\Delta$ and $\psi(x)$ becomes $\psi(x-\Delta)$, or $\psi_r(x) = \exp(-i\Delta P/h)\psi(x)$.

PINGBACKS

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