

SPINOR IN OSCILLATING MAGNETIC FIELD - PART 2

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.3, Part 2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In the first part of this article, we saw that a particle with spin placed in a precessing magnetic field can be analyzed by moving to a frame rotating with the same frequency as the field. In this rotating frame, the magnetic field is independent of time and looks like this:

$$\mathbf{B}_r = B\hat{\mathbf{x}}_r + \left(B_0 - \frac{\omega}{\gamma}\right)\hat{\mathbf{z}} \quad (1)$$

where $\hat{\mathbf{x}}_r$ is a unit vector along the rotating x axis. In this frame, the Schrödinger equation has the form

$$i\hbar \frac{\partial}{\partial t} |\psi_r(t)\rangle = -\gamma \mathbf{S} \cdot \mathbf{B}_r |\psi_r(t)\rangle \quad (2)$$

$$= [(\omega - \gamma B_0) S_z - \gamma B S_x] |\psi_r(t)\rangle \quad (3)$$

where $|\psi_r(t)\rangle$ is the state vector in the rotating frame, in the S_z basis. The Hamiltonian in the rotating frame is thus

$$H = (\omega - \gamma B_0) S_z - \gamma B S_x \quad (4)$$

$$= \frac{\hbar}{2} (\omega - \gamma B_0) \sigma_z - \frac{\hbar}{2} \gamma B \sigma_x \quad (5)$$

Given the initial state $|\psi_r(0)\rangle$ we can find the state at other times if we can find the propagator in the rotating frame

$$U_r(t) = e^{-iHt/\hbar} \quad (6)$$

The propagator is complicated by the fact that the Hamiltonian 5 contains two operators (σ_x and σ_z) that don't commute, so we can't split the exponential into the product of two simpler exponentials. However, if we expand the exponential in a power series, we see that it does actually have a fairly simple form. We have

$$e^{-iHt/\hbar} = e^{-i[(\omega - \gamma B_0)\sigma_z - \gamma B\sigma_x]t/2} \quad (7)$$

$$= e^{i[(\gamma B_0 - \omega)\sigma_z + \gamma B\sigma_x]t/2} \quad (8)$$

We can expand this in a power series, but first it's useful to introduce some shorthand. We have

$$\omega_0 \equiv \gamma B_0 \quad (9)$$

$$\omega_r \equiv \sqrt{(\gamma B_0 - \omega)^2 + \gamma^2 B^2} \quad (10)$$

$$= \sqrt{(\omega_0 - \omega)^2 + \gamma^2 B^2} \quad (11)$$

We get

$$e^{-iHt/\hbar} = I + \frac{it}{2} [(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x] + \quad (12)$$

$$- \frac{1}{2!} \frac{t^2}{2^2} [(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x]^2 + \quad (13)$$

$$- \frac{1}{3!} \frac{it^3}{2^3} [(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x]^3 + \dots \quad (14)$$

Consider the square term in the second line. Multiplying it out, we get

$$((\omega_0 - \omega)\sigma_z + \gamma B\sigma_x)^2 = (\omega_0 - \omega)^2 \sigma_z^2 + \gamma^2 B^2 \sigma_x^2 + \quad (15)$$

$$(\omega_0 - \omega)\gamma B(\sigma_z\sigma_x + \sigma_x\sigma_z) \quad (16)$$

Using a couple of identities for Pauli matrices:

$$\sigma_i^2 = I \quad (17)$$

$$[\sigma_z, \sigma_x]_{+} = 0 \quad (18)$$

we see that the last term vanishes and the first two terms can be combined, so we get

$$((\omega_0 - \omega)\sigma_z + \gamma B\sigma_x)^2 = [(\omega_0 - \omega)^2 + \gamma^2 B^2] I \quad (19)$$

$$= \omega_r^2 I \quad (20)$$

This simple form means that all higher terms in the power series 12 are easy to calculate. If we call the n th term in the series a_n , the terms with an even exponent are

$$a_{2n} = (-1)^n \frac{t^{2n}}{(2n)!2^{2n}} \omega_r^{2n} I \quad (21)$$

The $(-1)^n$ comes in because of the i in the exponent which gets raised to successively higher powers in the series. The series of even terms is therefore a cosine:

$$\sum_{n=0}^{\infty} a_{2n} = \cos \frac{\omega_r t}{2} I \quad (22)$$

For odd terms, we have

$$a_{2n+1} = (-1)^n i \omega_r^{2n} \frac{t^{2n+1}}{(2n+1)!2^{2n+1}} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x] \quad (23)$$

The series of odd terms comes out to

$$\sum_{n=0}^{\infty} a_{2n+1} = \frac{i}{\omega_r} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x] \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} \omega_r^{2n+1}}{(2n+1)!2^{2n+1}} \quad (24)$$

$$= \frac{i}{\omega_r} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x] \sin \frac{\omega_r t}{2} \quad (25)$$

We can therefore write out U as a matrix by using the Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (26)$$

$$U_r(t) = \begin{bmatrix} \cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} & \frac{i \gamma B}{\omega_r} \sin \frac{\omega_r t}{2} \\ \frac{i \gamma B}{\omega_r} \sin \frac{\omega_r t}{2} & \cos \frac{\omega_r t}{2} - \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \end{bmatrix} \quad (27)$$

To rotate this back to the lab frame, we apply the inverse rotation operator

$$e^{+i\omega t S_z / \hbar} = e^{i\omega t \sigma_z / 2} \quad (28)$$

$$= \begin{bmatrix} e^{i\omega t / 2} & 0 \\ 0 & e^{-i\omega t / 2} \end{bmatrix} \quad (29)$$

For a particle that starts in the spin up state

$$|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (30)$$

Since the spin z direction is also the axis of rotation for the rotating frame, we have (except for a phase factor that isn't observable physically):

$$|\psi_r(0)\rangle = e^{-i\omega t/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (31)$$

The general state at time t is

$$|\psi(t)\rangle = e^{i\omega t\sigma_z/2} U_r(t) |\psi_r(0)\rangle \quad (32)$$

$$= e^{-i\omega t/2} \begin{bmatrix} \cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{-i\omega t/2} \end{bmatrix} \quad (33)$$

In the case $\omega = \omega_0 = \gamma B_0$, we have $\omega_r = \gamma B$ from 11, so the state vector becomes

$$|\psi(t)\rangle = e^{-i\omega t/2} \begin{bmatrix} \cos \frac{\gamma B t}{2} e^{i\omega t/2} \\ i \sin \frac{\gamma B t}{2} e^{-i\omega t/2} \end{bmatrix} \quad (34)$$

If we compare this to the eigenvector $|\hat{n}+\rangle$ for spin up along a general direction given by the spherical angles θ and ϕ , which is

$$|\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (35)$$

we see that, apart from the extra i in the sine term, the state $|\psi(t)\rangle$ is the spin-up state for polar angles $\theta = \gamma B t$, $\phi = -\omega t$. The probability of finding an up or down state is

$$P_{up} = \left| \cos \frac{\gamma B t}{2} e^{i\omega t/2} \right|^2 = \cos^2 \frac{\gamma B t}{2} \quad (36)$$

$$P_{down} = \left| i \sin \frac{\gamma B t}{2} e^{-i\omega t/2} \right|^2 = \sin^2 \frac{\gamma B t}{2} \quad (37)$$

The spin oscillates between a pure up state when $\gamma B t/2$ is a multiple of π to a pure down state when $\gamma B t/2$ is an odd multiple of $\frac{\pi}{2}$.

Finally, we can check that $\langle \mu_z(t) \rangle$ agrees with the classical result

$$\mu_z(t) = \mu_z(0) \left[\frac{(\omega_0 - \omega)^2}{\gamma^2 B^2 + (\omega_0 - \omega)^2} + \frac{\gamma^2 B^2 \cos \omega t}{\gamma^2 B^2 + (\omega_0 - \omega)^2} \right] \quad (38)$$

To find $\langle \mu_z(t) \rangle$ we evaluate as follows.

$$\langle \mu_z(t) \rangle = \langle \psi(t) | \mu_z | \psi(t) \rangle \quad (39)$$

$$= \gamma \langle \psi(t) | S_z | \psi(t) \rangle \quad (40)$$

$$= \frac{\gamma \hbar}{2} \langle \psi(t) | \sigma_z | \psi(t) \rangle \quad (41)$$

$$= \frac{\gamma \hbar}{2} \left[\left(\cos \frac{\omega_r t}{2} - \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \right) e^{-i\omega t/2} \quad -\frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{i\omega t/2} \right] \times \quad (42)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \left(\cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \right) e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{-i\omega t/2} \end{bmatrix} \quad (43)$$

We introduce shorthand for the trig functions:

$$c \equiv \cos \frac{\omega_r t}{2} \quad (44)$$

$$s \equiv \sin \frac{\omega_r t}{2} \quad (45)$$

Then we have (note that complex exponentials cancel out):

$$\frac{2}{\gamma \hbar} \langle \mu_z(t) \rangle = \left[\left[c - \frac{\omega_0 - \omega}{\omega_r} i s \right] e^{-i\omega t/2} \quad -\frac{i\gamma B}{\omega_r} s e^{i\omega t/2} \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \left[c + \frac{\omega_0 - \omega}{\omega_r} i s \right] e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r} s e^{-i\omega t/2} \end{bmatrix} \quad (46)$$

$$\left[c - \frac{\omega_0 - \omega}{\omega_r} i s \quad -\frac{i\gamma B}{\omega_r} s \right] \begin{bmatrix} c + \frac{\omega_0 - \omega}{\omega_r} i s \\ -\frac{i\gamma B}{\omega_r} s \end{bmatrix} \quad (47)$$

$$= c^2 + \left(\frac{\omega_0 - \omega}{\omega_r} \right)^2 s^2 - \left(\frac{\gamma B}{\omega_r} \right)^2 s^2 \quad (48)$$

$$= \frac{1}{\omega_r^2} \left[\omega_r^2 c^2 + \left((\omega_0 - \omega)^2 - \gamma^2 B^2 \right) s^2 \right] \quad (49)$$

$$= \frac{1}{\omega_r^2} \left[\left((\omega_0 - \omega)^2 + \gamma^2 B^2 \right) c^2 + \left((\omega_0 - \omega)^2 - \gamma^2 B^2 \right) s^2 \right] \quad (50)$$

$$= \frac{1}{\omega_r^2} \left((\omega_0 - \omega)^2 + \gamma^2 B^2 (c^2 - s^2) \right) \quad (51)$$

where we used 11 in the fourth line.

Using the trig identity

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad (52)$$

we see that

$$c^2 - s^2 = \cos^2 \frac{\omega_r t}{2} - \sin^2 \frac{\omega_r t}{2} = \cos \omega_r t \quad (53)$$

So we have

$$\langle \mu_z(t) \rangle = \frac{\gamma \hbar (\omega_0 - \omega)^2 + \gamma^2 B^2 \cos \omega_r t}{2 \omega_r^2} \quad (54)$$

$$= \frac{\gamma \hbar (\omega_0 - \omega)^2 + \gamma^2 B^2 \cos \omega_r t}{2 (\omega_0 - \omega)^2 + \gamma^2 B^2} \quad (55)$$

This agrees with 38 provided $\mu_z(0) = \frac{\gamma \hbar}{2}$, which is true, since the magnitude of the magnetic moment is $\frac{\gamma \hbar}{2}$ and it starts in the spin up position so $\mu_z(0) = \frac{\gamma \hbar}{2}$.