SPINOR IN OSCILLATING MAGNETIC FIELD - PART 2

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.3, Part 2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In the first part of this article, we saw that a particle with spin placed in a precessing magnetic field can be analyzed by moving to a frame rotating with the same frequency as the field. In this rotating frame, the magnetic field is independent of time and looks like this:

$$\mathbf{B}_r = B\hat{\mathbf{x}}_r + \left(B_0 - \frac{\omega}{\gamma}\right)\hat{\mathbf{z}} \tag{1}$$

where $\hat{\mathbf{x}}_r$ is a unit vector along the rotating x axis. In this frame, the Schrödinger equation has the form

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\psi_r(t)\right\rangle &= -\gamma\mathbf{S}\cdot\mathbf{B}_r\left|\psi_r(t)\right\rangle & (2)\\ &= \left[(\omega-\gamma B_0)S_z - \gamma B S_x\right]\left|\psi_r(t)\right\rangle & (3)\end{aligned}$$

where $\left|\psi_r(t)\right\rangle$ is the state vector in the rotating frame, in the S_z basis. The Hamiltonian in the rotating frame is thus

$$\begin{aligned} H &= (\omega-\gamma B_0)S_z - \gamma B S_x & (4)\\ &= \frac{\hbar}{2}(\omega-\gamma B_0)\sigma_z - \frac{\hbar}{2}\gamma B\sigma_x & (5)\end{aligned}$$

Given the initial state $\left|\psi_r(0)\right\rangle$ we can find the state at other times if we can find the propagator in the rotating frame

$$U_r(t) = e^{-iHt/\hbar} \tag{6}$$

The propagator is complicated by the fact that the Hamiltonian 5 contains two operators (σ_x and σ_z) that don't commute, so we can't split the exponential into the product of two simpler exponentials. However, if we expand the exponential in a power series, we see that it does actually have a fairly simple form. We have

$$e^{-iHt/\hbar} = e^{-i[(\omega-\gamma B_0)\sigma_z-\gamma B\sigma_x]t/2} \tag{7}$$
$$= e^{i[(\gamma B_0-\omega)\sigma_z+\gamma B\sigma_x]t/2} \tag{8}$$

We can expand this in a power series, but first it's useful to introduce some shorthand. We have

$$\omega_0 \equiv \gamma B_0 \tag{9}$$
$$\omega_r \equiv \sqrt{(\gamma B_0-\omega)^2+\gamma^2 B^2} \tag{10}$$
$$= \sqrt{(\omega_0-\omega)^2+\gamma^2 B^2} \tag{11}$$

We get

$$e^{-iHt/\hbar} = I+\frac{it}{2}\left[(\omega_0-\omega)\sigma_z+\gamma B\sigma_x\right]+ \tag{12}$$
$$-\frac{1}{2!}\frac{t^2}{2^2}\left(\left[(\omega_0-\omega)\sigma_z+\gamma B\sigma_x\right]\right)^2+ \tag{13}$$
$$-\frac{1}{3!}\frac{it^3}{2^3}\left(\left[(\omega_0-\omega)\sigma_z+\gamma B\sigma_x\right]\right)^3+\ldots \tag{14}$$

Consider the square term in the second line. Multiplying it out, we get

$$((\omega_0-\omega)\sigma_z+\gamma B\sigma_x)^2 = (\omega_0-\omega)^2\sigma_z^2+\gamma^2 B^2\sigma_x^2+ \tag{15}$$
$$(\omega_0-\omega)\gamma B(\sigma_z\sigma_x+\sigma_x\sigma_z) \tag{16}$$

Using a couple of identities for Pauli matrices:

$$\sigma_i^2 = I \tag{17}$$
$$[\sigma_z,\sigma_x]_+ = 0 \tag{18}$$

we see that the last term vanishes and the first two terms can be combined, so we get

$$((\omega_0-\omega)\sigma_z+\gamma B\sigma_x)^2 = \left[(\omega_0-\omega)^2+\gamma^2 B^2\right] I \tag{19}$$
$$= \omega_r^2 I \tag{20}$$

This simple form means that all higher terms in the power series 12 are easy to calculate. If we call the nth term in the series a_n, the terms with an even exponent are

$$a_{2n}=(-1)^{n}\frac{t^{2n}}{(2n)!2^{2n}}\omega_{r}^{2n}I \tag{21}$$

The $(-1)^n$ comes in because of the i in the exponent which gets raised to successively higher powers in the series. The series of even terms is therefore a cosine:

$$\sum_{n=0}^{\infty}a_{2n}=\cos\frac{\omega_{r}t}{2}I \tag{22}$$

For odd terms, we have

$$a_{2n+1}=(-1)^{n}i\omega_{r}^{2n}\frac{t^{2n+1}}{(2n+1)!2^{2n+1}}\left[(\omega_{0}-\omega)\sigma_{z}+\gamma B\sigma_{x}\right] \tag{23}$$

The series of odd terms comes out to

$$\sum_{n=0}^{\infty}a_{2n+1} = \frac{i}{\omega_{r}}\left[(\omega_{0}-\omega)\sigma_{z}+\gamma B\sigma_{x}\right]\sum_{n=0}^{\infty}\frac{(-1)^{n}t^{2n+1}\omega_{r}^{2n+1}}{(2n+1)!2^{2n+1}} \tag{24}$$

$$= \frac{i}{\omega_{r}}\left[(\omega_{0}-\omega)\sigma_{z}+\gamma B\sigma_{x}\right]\sin\frac{\omega_{r}t}{2} \tag{25}$$

We can therefore write out U as a matrix by using the Pauli matrices:

$$\sigma_{x}=\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right];\quad\sigma_{z}=\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right] \tag{26}$$

$$U_{r}(t)=\left[\begin{array}{cc}\cos\frac{\omega_{r}t}{2}+\frac{\omega_{0}-\omega}{\omega_{r}}i\sin\frac{\omega_{r}t}{2} & \frac{i\gamma B}{\omega_{r}}\sin\frac{\omega_{r}t}{2}\\ \frac{i\gamma B}{\omega_{r}}\sin\frac{\omega_{r}t}{2} & \cos\frac{\omega_{r}t}{2}-\frac{\omega_{0}-\omega}{\omega_{r}}i\sin\frac{\omega_{r}t}{2}\end{array}\right] \tag{27}$$

To rotate this back to the lab frame, we apply the inverse rotation operator

$$e^{+i\omega tS_{z}/\hbar} = e^{i\omega t\sigma_{z}/2} \tag{28}$$

$$= \left[\begin{array}{cc}e^{i\omega t/2} & 0\\ 0 & e^{-i\omega t/2}\end{array}\right] \tag{29}$$

For a particle that starts in the spin up state

$$|\psi(0)\rangle=\left[\begin{array}{c}1\\ 0\end{array}\right] \tag{30}$$

Since the spin z direction is also the axis of rotation for the rotating frame, we have (except for a phase factor that isn't observable physically):

$$|\psi_r(0)\rangle = e^{-i\omega t/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{31}$$

The general state at time t is

$$|\psi(t)\rangle = e^{i\omega t\sigma_z/2}U_r(t)|\psi_r(0)\rangle \tag{32}$$

$$= e^{-i\omega t/2}\begin{bmatrix} \left[\cos\frac{\omega_r t}{2} + \frac{\omega_0-\omega}{\omega_r} i\sin\frac{\omega_r t}{2}\right]e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r}\sin\frac{\omega_r t}{2}e^{-i\omega t/2} \end{bmatrix} \tag{33}$$

In the case $\omega = \omega_0 = \gamma B_0$, we have $\omega_r = \gamma B$ from 11, so the state vector becomes

$$|\psi(t)\rangle = e^{-i\omega t/2}\begin{bmatrix} \cos\frac{\gamma Bt}{2}e^{i\omega t/2} \\ i\sin\frac{\gamma Bt}{2}e^{-i\omega t/2} \end{bmatrix} \tag{34}$$

If we compare this to the eigenvector $|\hat{n}+\rangle$ for spin up along a general direction given by the spherical angles θ and ϕ, which is

$$|\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{35}$$

we see that, apart from the extra i in the sine term, the state $|\psi(t)\rangle$ is the spin-up state for polar angles $\theta = \gamma Bt$, $\phi = -\omega t$. The probability of finding an up or down state is

$$P_{up} = \left|\cos\frac{\gamma Bt}{2}e^{i\omega t/2}\right|^2 = \cos^2\frac{\gamma Bt}{2} \tag{36}$$

$$P_{down} = \left|i\sin\frac{\gamma Bt}{2}e^{-i\omega t/2}\right|^2 = \sin^2\frac{\gamma Bt}{2} \tag{37}$$

The spin oscillates between a pure up state when $\gamma Bt/2$ is a multiple of π to a pure down state when $\gamma Bt/2$ is an odd multiple of $\frac{\pi}{2}$.

Finally, we can check that $\langle\mu_z(t)\rangle$ agrees with the classical result

$$\mu_z(t) = \mu_z(0)\left[\frac{(\omega_0-\omega)^2}{\gamma^2B^2+(\omega_0-\omega)^2} + \frac{\gamma^2B^2\cos\omega t}{\gamma^2B^2+(\omega_0-\omega)^2}\right] \tag{38}$$

To find $\langle\mu_z(t)\rangle$ we evaluate as follows.

$$\begin{aligned}
\langle\mu_z(t)\rangle &= \langle\psi(t)|\mu_z|\psi(t)\rangle && (39)\\
&= \gamma\langle\psi(t)|S_z|\psi(t)\rangle && (40)\\
&= \frac{\gamma\hbar}{2}\langle\psi(t)|\sigma_z|\psi(t)\rangle && (41)\\
&= \frac{\gamma\hbar}{2}\left[\begin{array}{cc}\left(\cos\frac{\omega_r t}{2}-\frac{\omega_0-\omega}{\omega_r}i\sin\frac{\omega_r t}{2}\right)e^{-i\omega t/2} & -\frac{i\gamma B}{\omega_r}\sin\frac{\omega_r t}{2}e^{i\omega t/2}\end{array}\right]\times && (42)\\
&\left[\begin{array}{cc}1&0\\0&-1\end{array}\right]\left[\begin{array}{c}\left(\cos\frac{\omega_r t}{2}+\frac{\omega_0-\omega}{\omega_r}i\sin\frac{\omega_r t}{2}\right)e^{i\omega t/2}\\ \frac{i\gamma B}{\omega_r}\sin\frac{\omega_r t}{2}e^{-i\omega t/2}\end{array}\right] && (43)
\end{aligned}$$

We introduce shorthand for the trig functions:

$$c \equiv \cos\frac{\omega_r t}{2} \quad (44)$$

$$s \equiv \sin\frac{\omega_r t}{2} \quad (45)$$

Then we have (note that complex exponentials cancel out):

$$\begin{aligned}
\frac{2}{\gamma\hbar}\langle\mu_z(t)\rangle &= \left[\begin{array}{cc}\left[c-\frac{\omega_0-\omega}{\omega_r}is\right]e^{-i\omega t/2} & -\frac{i\gamma B}{\omega_r}se^{i\omega t/2}\end{array}\right]\left[\begin{array}{cc}1&0\\0&-1\end{array}\right]\left[\begin{array}{c}\left[c+\frac{\omega_0-\omega}{\omega_r}is\right]e^{i\omega t/2}\\ \frac{i\gamma B}{\omega_r}se^{-i\omega t/2}\end{array}\right] && (46)\\
&\left[\begin{array}{cc}c-\frac{\omega_0-\omega}{\omega_r}is & -\frac{i\gamma B}{\omega_r}s\end{array}\right]\left[\begin{array}{c}c+\frac{\omega_0-\omega}{\omega_r}is\\ -\frac{i\gamma B}{\omega_r}s\end{array}\right] && (47)\\
&= c^2+\left(\frac{\omega_0-\omega}{\omega_r}\right)^2 s^2-\left(\frac{\gamma B}{\omega_r}\right)^2 s^2 && (48)\\
&= \frac{1}{\omega_r^2}\left[\omega_r^2c^2+\left((\omega_0-\omega)^2-\gamma^2B^2\right)s^2\right] && (49)\\
&= \frac{1}{\omega_r^2}\left[\left((\omega_0-\omega)^2+\gamma^2B^2\right)c^2+\left((\omega_0-\omega)^2-\gamma^2B^2\right)s^2\right] && (50)\\
&= \frac{1}{\omega_r^2}\left((\omega_0-\omega)^2+\gamma^2B^2\left(c^2-s^2\right)\right) && (51)
\end{aligned}$$

where we used 11 in the fourth line.
Using the trig identity

$$\cos 2\theta = \cos^2\theta-\sin^2\theta \quad (52)$$

we see that

$$c^2 - s^2 = \cos^2\frac{\omega_r t}{2} - \sin^2\frac{\omega_r t}{2} = \cos\omega_r t \tag{53}$$

So we have

$$\langle\mu_z(t)\rangle = \frac{\gamma\hbar}{2}\frac{(\omega_0 - \omega)^2 + \gamma^2 B^2\cos\omega_r t}{\omega_r^2} \tag{54}$$

$$= \frac{\gamma\hbar}{2}\frac{(\omega_0 - \omega)^2 + \gamma^2 B^2\cos\omega_r t}{(\omega_0 - \omega)^2 + \gamma^2 B^2} \tag{55}$$

This agrees with 38 provided $\mu_z(0) = \frac{\gamma\hbar}{2}$, which is true, since the magnitude of the magnetic moment is $\frac{\gamma\hbar}{2}$ and it starts in the spin up position so $\mu_z(0) = \frac{\gamma\hbar}{2}$.