

SPINOR IN OSCILLATING MAGNETIC FIELD - PART 2

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.3, Part 2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In the first part of this article, we saw that a particle with spin placed in a precessing magnetic field can be analyzed by moving to a frame rotating with the same frequency as the field. In this rotating frame, the magnetic field is independent of time and looks like this:

$$(0.1) \quad \mathbf{B}_r = B\hat{\mathbf{x}}_r + \left(B_0 - \frac{\omega}{\gamma}\right)\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}_r$ is a unit vector along the rotating x axis. In this frame, the Schrödinger equation has the form

$$(0.2) \quad i\hbar \frac{\partial}{\partial t} |\psi_r(t)\rangle = -\gamma \mathbf{S} \cdot \mathbf{B}_r |\psi_r(t)\rangle$$

$$(0.3) \quad = [(\omega - \gamma B_0) S_z - \gamma B S_x] |\psi_r(t)\rangle$$

where $|\psi_r(t)\rangle$ is the state vector in the rotating frame, in the S_z basis. The Hamiltonian in the rotating frame is thus

$$(0.4) \quad H = (\omega - \gamma B_0) S_z - \gamma B S_x$$

$$(0.5) \quad = \frac{\hbar}{2} (\omega - \gamma B_0) \sigma_z - \frac{\hbar}{2} \gamma B \sigma_x$$

Given the initial state $|\psi_r(0)\rangle$ we can find the state at other times if we can find the propagator in the rotating frame

$$(0.6) \quad U_r(t) = e^{-iHt/\hbar}$$

The propagator is complicated by the fact that the Hamiltonian 0.5 contains two operators (σ_x and σ_z) that don't commute, so we can't split the exponential into the product of two simpler exponentials. However, if we

expand the exponential in a power series, we see that it does actually have a fairly simple form. We have

$$(0.7) \quad e^{-iHt/\hbar} = e^{-i[(\omega - \gamma B_0)\sigma_z - \gamma B\sigma_x]t/2}$$

$$(0.8) \quad = e^{i[(\gamma B_0 - \omega)\sigma_z + \gamma B\sigma_x]t/2}$$

We can expand this in a power series, but first it's useful to introduce some shorthand. We have

$$(0.9) \quad \omega_0 \equiv \gamma B_0$$

$$(0.10) \quad \omega_r \equiv \sqrt{(\gamma B_0 - \omega)^2 + \gamma^2 B^2}$$

$$(0.11) \quad = \sqrt{(\omega_0 - \omega)^2 + \gamma^2 B^2}$$

We get

$$(0.12) \quad e^{-iHt/\hbar} = I + \frac{it}{2} [(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x] +$$

$$(0.13) \quad - \frac{1}{2!} \frac{t^2}{2^2} ([(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x])^2 +$$

$$(0.14) \quad - \frac{1}{3!} \frac{it^3}{2^3} ([(\omega_0 - \omega)\sigma_z + \gamma B\sigma_x])^3 + \dots$$

Consider the square term in the second line. Multiplying it out, we get

$$(0.15) \quad ((\omega_0 - \omega)\sigma_z + \gamma B\sigma_x)^2 = (\omega_0 - \omega)^2 \sigma_z^2 + \gamma^2 B^2 \sigma_x^2 +$$

$$(0.16) \quad (\omega_0 - \omega)\gamma B(\sigma_z\sigma_x + \sigma_x\sigma_z)$$

Using a couple of identities for Pauli matrices:

$$(0.17) \quad \sigma_i^2 = I$$

$$(0.18) \quad [\sigma_z, \sigma_x]_{+} = 0$$

we see that the last term vanishes and the first two terms can be combined, so we get

$$(0.19) \quad ((\omega_0 - \omega)\sigma_z + \gamma B\sigma_x)^2 = [(\omega_0 - \omega)^2 + \gamma^2 B^2] I$$

$$(0.20) \quad = \omega_r^2 I$$

This simple form means that all higher terms in the power series 0.12 are easy to calculate. If we call the n th term in the series a_n , the terms with an even exponent are

$$(0.21) \quad a_{2n} = (-1)^n \frac{t^{2n}}{(2n)!2^{2n}} \omega_r^{2n} I$$

The $(-1)^n$ comes in because of the i in the exponent which gets raised to successively higher powers in the series. The series of even terms is therefore a cosine:

$$(0.22) \quad \sum_{n=0}^{\infty} a_{2n} = \cos \frac{\omega_r t}{2} I$$

For odd terms, we have

$$(0.23) \quad a_{2n+1} = (-1)^n i \omega_r^{2n} \frac{t^{2n+1}}{(2n+1)!2^{2n+1}} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x]$$

The series of odd terms comes out to

$$(0.24) \quad \sum_{n=0}^{\infty} a_{2n+1} = \frac{i}{\omega_r} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x] \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} \omega_r^{2n+1}}{(2n+1)!2^{2n+1}}$$

$$(0.25) \quad = \frac{i}{\omega_r} [(\omega_0 - \omega) \sigma_z + \gamma B \sigma_x] \sin \frac{\omega_r t}{2}$$

We can therefore write out U as a matrix by using the Pauli matrices:

$$(0.26) \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(0.27) \quad U_r(t) = \begin{bmatrix} \cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} & \frac{i \gamma B}{\omega_r} \sin \frac{\omega_r t}{2} \\ \frac{i \gamma B}{\omega_r} \sin \frac{\omega_r t}{2} & \cos \frac{\omega_r t}{2} - \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \end{bmatrix}$$

To rotate this back to the lab frame, we apply the inverse rotation operator

$$(0.28) \quad e^{+i\omega t S_z / \hbar} = e^{i\omega t \sigma_z / 2}$$

$$(0.29) \quad = \begin{bmatrix} e^{i\omega t / 2} & 0 \\ 0 & e^{-i\omega t / 2} \end{bmatrix}$$

For a particle that starts in the spin up state

$$(0.30) \quad |\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the spin z direction is also the axis of rotation for the rotating frame, we have (except for a phase factor that isn't observable physically):

$$(0.31) \quad |\psi_r(0)\rangle = e^{-i\omega t/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general state at time t is

$$(0.32) \quad |\psi(t)\rangle = e^{i\omega t\sigma_z/2} U_r(t) |\psi_r(0)\rangle$$

$$(0.33) \quad = e^{-i\omega t/2} \begin{bmatrix} \left[\cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \right] e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{-i\omega t/2} \end{bmatrix}$$

In the case $\omega = \omega_0 = \gamma B_0$, we have $\omega_r = \gamma B$ from 0.11, so the state vector becomes

$$(0.34) \quad |\psi(t)\rangle = e^{-i\omega t/2} \begin{bmatrix} \cos \frac{\gamma B t}{2} e^{i\omega t/2} \\ i \sin \frac{\gamma B t}{2} e^{-i\omega t/2} \end{bmatrix}$$

If we compare this to the eigenvector $|\hat{n}+\rangle$ for spin up along a general direction given by the spherical angles θ and ϕ , which is

$$(0.35) \quad |\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

we see that, apart from the extra i in the sine term, the state $|\psi(t)\rangle$ is the spin-up state for polar angles $\theta = \gamma B t$, $\phi = -\omega t$. The probability of finding an up or down state is

$$(0.36) \quad P_{up} = \left| \cos \frac{\gamma B t}{2} e^{i\omega t/2} \right|^2 = \cos^2 \frac{\gamma B t}{2}$$

$$(0.37) \quad P_{down} = \left| i \sin \frac{\gamma B t}{2} e^{-i\omega t/2} \right|^2 = \sin^2 \frac{\gamma B t}{2}$$

The spin oscillates between a pure up state when $\gamma B t/2$ is a multiple of π to a pure down state when $\gamma B t/2$ is an odd multiple of $\frac{\pi}{2}$.

Finally, we can check that $\langle \mu_z(t) \rangle$ agrees with the classical result

$$(0.38) \quad \mu_z(t) = \mu_z(0) \left[\frac{(\omega_0 - \omega)^2}{\gamma^2 B^2 + (\omega_0 - \omega)^2} + \frac{\gamma^2 B^2 \cos \omega t}{\gamma^2 B^2 + (\omega_0 - \omega)^2} \right]$$

To find $\langle \mu_z(t) \rangle$ we evaluate as follows.

$$(0.39)$$

$$\langle \mu_z(t) \rangle = \langle \psi(t) | \mu_z | \psi(t) \rangle$$

$$(0.40) \quad = \gamma \langle \psi(t) | S_z | \psi(t) \rangle$$

$$(0.41) \quad = \frac{\gamma \hbar}{2} \langle \psi(t) | \sigma_z | \psi(t) \rangle$$

$$(0.42) \quad = \frac{\gamma \hbar}{2} \left[\left(\cos \frac{\omega_r t}{2} - \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \right) e^{-i\omega t/2} \quad -\frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{i\omega t/2} \right] \times$$

$$(0.43) \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \left(\cos \frac{\omega_r t}{2} + \frac{\omega_0 - \omega}{\omega_r} i \sin \frac{\omega_r t}{2} \right) e^{i\omega t/2} \\ \frac{i\gamma B}{\omega_r} \sin \frac{\omega_r t}{2} e^{-i\omega t/2} \end{bmatrix}$$

We introduce shorthand for the trig functions:

$$(0.44) \quad c \equiv \cos \frac{\omega_r t}{2}$$

$$(0.45) \quad s \equiv \sin \frac{\omega_r t}{2}$$

Then we have (note that complex exponentials cancel out):

(0.46)

$$\frac{2}{\gamma\hbar} \langle \mu_z(t) \rangle = \left[\left[c - \frac{\omega_0 - \omega}{\omega_r} is \right] e^{-i\omega t/2} \quad -\frac{i\gamma B}{\omega_r} s e^{i\omega t/2} \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left[\begin{bmatrix} c + \frac{\omega_0 - \omega}{\omega_r} is \\ \frac{i\gamma B}{\omega_r} s e^{-i\omega t/2} \end{bmatrix} \right]$$

(0.47)

$$\left[c - \frac{\omega_0 - \omega}{\omega_r} is \quad -\frac{i\gamma B}{\omega_r} s \right] \begin{bmatrix} c + \frac{\omega_0 - \omega}{\omega_r} is \\ -\frac{i\gamma B}{\omega_r} s \end{bmatrix}$$

(0.48)

$$= c^2 + \left(\frac{\omega_0 - \omega}{\omega_r} \right)^2 s^2 - \left(\frac{\gamma B}{\omega_r} \right)^2 s^2$$

(0.49)

$$= \frac{1}{\omega_r^2} \left[\omega_r^2 c^2 + \left((\omega_0 - \omega)^2 - \gamma^2 B^2 \right) s^2 \right]$$

(0.50)

$$= \frac{1}{\omega_r^2} \left[\left((\omega_0 - \omega)^2 + \gamma^2 B^2 \right) c^2 + \left((\omega_0 - \omega)^2 - \gamma^2 B^2 \right) s^2 \right]$$

(0.51)

$$= \frac{1}{\omega_r^2} \left((\omega_0 - \omega)^2 + \gamma^2 B^2 (c^2 - s^2) \right)$$

where we used 0.11 in the fourth line.

Using the trig identity

$$(0.52) \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

we see that

$$(0.53) \quad c^2 - s^2 = \cos^2 \frac{\omega_r t}{2} - \sin^2 \frac{\omega_r t}{2} = \cos \omega_r t$$

So we have

$$(0.54) \quad \langle \mu_z(t) \rangle = \frac{\gamma\hbar (\omega_0 - \omega)^2 + \gamma^2 B^2 \cos \omega_r t}{2 \omega_r^2}$$

$$(0.55) \quad = \frac{\gamma\hbar (\omega_0 - \omega)^2 + \gamma^2 B^2 \cos \omega_r t}{2 (\omega_0 - \omega)^2 + \gamma^2 B^2}$$

This agrees with 0.38 provided $\mu_z(0) = \frac{\gamma\hbar}{2}$, which is true, since the magnitude of the magnetic moment is $\frac{\gamma\hbar}{2}$ and it starts in the spin up position so $\mu_z(0) = \frac{\gamma\hbar}{2}$.