EFFECTIVE MAGNETIC FIELD IN ROTATING FRAME - AXIS OF ROTATION NOT PARALLEL TO FIELD

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.4.5.

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Returning to classical physics for this post, we recall that if we have a magnetic moment μ placed in a constant magnetic field $\mathbf{B}_0$, then if we move to a frame of reference that rotates with frequency ω which is parallel to $\mathbf{B}_0$, the effective magnetic field as seen in the rotating frame is

$$\mathbf{B}_r = \mathbf{B}_0 + \frac{\omega}{\gamma} \tag{1}$$

where γ is the gyromagnetic ratio. However, suppose the rotating frame has an axis of rotation that is not parallel to $\mathbf{B}_0$, that is ω is not parallel to $\mathbf{B}_0$. In this case, we have the following situation.

Suppose we have some arbitrary vector $\mathbf{V}$ which changes by $\Delta\mathbf{V}$ in time interval Δt, as viewed in the non-rotating frame. As we've seen earlier, if a vector $\mathbf{r}$ that makes an angle α with the axis of rotation is rotated through an angle $\Delta\theta$ where the direction of $\Delta\theta$ is the axis of rotation, then the change in $\mathbf{r}$ is given by

$$\mathbf{r} \rightarrow \mathbf{r} + (\delta\theta) \times \mathbf{r} \tag{2}$$

In the rotating frame, in the time interval Δt, the vector changes due to two separate effects: the change $\Delta\mathbf{V}$ that occurs in the lab frame *plus* the change due to the angle $\delta\theta = -\omega\Delta t$ that occurs due to the frame's rotation. (The minus sign arises because if the frame is rotating counterclockwise, objects in the lab frame appear to be rotating clockwise as seen in the rotating frame.) If we take the vector to be at the same position in both frames at time t, then our job is to find the relation between the changes to $\mathbf{V}$ that occur in the two frames after interval Δt.

In the inertial (lab) frame, we have

$$\mathbf{V}(t + \Delta t) = \mathbf{V}(t) + \Delta\mathbf{V} \tag{3}$$

In the rotating frame, we have

$$\begin{aligned} \mathbf{V}_r(t+\Delta t) &= \mathbf{V}_r(t)+\Delta\mathbf{V}_r && (4)\\ &= \mathbf{V}(t)+\Delta\mathbf{V}_r && (5)\end{aligned}$$

where the last line follows because the two vectors are identical at the initial time t.

Now by applying 2 to a rotation angle of $\delta\theta = -\omega\Delta t$ we have

$$\begin{aligned} \mathbf{V}_r(t+\Delta t) &= \mathbf{V}(t+\Delta t)-\omega\times\mathbf{V}(t+\Delta t)\Delta t && (6)\\ &= \mathbf{V}(t)+\Delta\mathbf{V}-\omega\times[\mathbf{V}(t)+\Delta\mathbf{V}]\Delta t && (7)\\ &= \mathbf{V}_r(t)+\Delta\mathbf{V}-\omega\times\mathbf{V}(t)\Delta t+\mathcal{O}\left((\Delta t)^2\right) && (8)\\ \Delta\mathbf{V}_r &= \mathbf{V}_r(t+\Delta t)-\mathbf{V}_r(t) && (9)\\ &= \Delta\mathbf{V}-\omega\times\mathbf{V}(t)\Delta t && (10)\end{aligned}$$

where the last line drops higher order terms. Dividing through by Δt and taking the limit we get

$$\frac{d\mathbf{V}_r}{dt}=\frac{d\mathbf{V}}{dt}-\omega\times\mathbf{V} \tag{11}$$

If we now apply this to the precession of magnetic moments, we begin with the relation between torque $\mathbf{T}$ and angular momentum ℓ:

$$\mathbf{T}=\frac{d\ell}{dt}=\gamma\ell\times\mathbf{B}_0 \tag{12}$$

In the rotating frame, we have

$$\begin{aligned} \frac{d\ell_r}{dt} &= \frac{d\ell}{dt}-\omega\times\ell && (13)\\ &= \gamma\ell\times\mathbf{B}_0+\ell\times\omega && (14)\\ &= \gamma\ell\times\left(\mathbf{B}_0+\frac{\omega}{\gamma}\right) && (15)\end{aligned}$$

Thus the effective field in the rotating frame is again

$$\mathbf{B}_r=\mathbf{B}_0+\frac{\omega}{\gamma} \tag{16}$$

and this applies even if $\mathbf{B}_0$ and ω are not parallel.