STERN-GERLACH EXPERIMENT

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 14.5.3 - 14.5.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The Stern-Gerlach experiment is the classic experiment that revealed the existence of electron spin. Shankar describes the ideas behind the experiment at the end of Chapter 14, so we'll just summarize them here.

The idea is to pass a beam of particles possessing magnetic moments through a non-uniform magnetic field $\mathbf{B}$. The $\mathbf{B}$ field has a gradient along the z axis and the beam of particles is fired into this field along the y axis. The force exerted by this non-uniform field on a particle with magnetic moment μ is given by

$$\mathbf{F}=-\nabla H=\mu_{z}\frac{\partial B_{z}}{\partial z}\hat{\mathbf{z}} \tag{1}$$

(see Shankar for the details of this calculation). If the magnetic moments μ_z have a continuous spread (as would be expected classically), then the force ranges continuously and we'd expect to see the particles smeared out over a uniform strip on the detector. What is actually observed is that the particles are deflected in discrete intervals, so we get a series of dots on the detector rather than a continuous line. This is explained by the fact that the magnetic moment (arising either from spin or orbital angular momentum) is quantized.

Here are some examples of what the experiment would reveal.

If we start with a beam of spin-$\frac{1}{2}$ particles (such as electrons or neutral hydrogen atoms where we can neglect the magnetic moment of the proton) then, because the possible values of spin are $\pm\frac{\hbar}{2}$, we see the incident beam split into two beams. If we block the lower beam, and allow the upper beam through, then we have a beam containing particles all with spin up, or $+\frac{\hbar}{2}$. If we pass this beam into a second appartus with a $\mathbf{B}$ field along the x axis (so the field gradient is at an angle of $\frac{\pi}{2}$ relative to the first apparatus), this second apparatus will also split the beam into two sub-beams. Because the spin operators S_z and S_x don't commute, we can't measure the spin components in both directions simultaneously so, as far as the second apparatus is concerned, the x component of spin is unknown and could be either $\pm\frac{\hbar}{2}$. If we block the lower beam from the second apparatus, what fraction of the particles will get through?

To answer this, recall that the eigenspinors for the x direction in the basis of z spinors are

$$\chi_{+}^{(x)}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\1\end{array}\right) \tag{2}$$

$$\chi_{-}^{(x)}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\-1\end{array}\right) \tag{3}$$

Thus a particle with x spin $+\frac{\hbar}{2}$ is equally likely to be measured with a z spin of $\pm\frac{\hbar}{2}$. We can apply this argument in reverse by swapping the definitions of the x and z axes, so that if we send a beam of particles with z spin $+\frac{\hbar}{2}$ into the second apparatus, then a particle is equally likely to have a spin of $\pm\frac{\hbar}{2}$ in the x direction. In other words, on average, half the particles being fed into the second apparatus will go into the latter's 'up' beam and half into its 'down' beam. Combining the two apparatuses, we'd expect on average $\frac{1}{4}$ of the incident particles to emerge in the upper beam of the second apparatus.

Another way of saying this is that if we arrange a sequence of apparatuses where each apparatus is rotated by $\frac{\pi}{2}$ relative to its predecessor and one of the exit beams in each case is blocked, then the number of particles getting through each apparatus is half the number that entered it.

Now suppose we return to the case where the first apparatus transmits only spin z of $+\frac{\hbar}{2}$ but the second (aligned along the x axis) transmits everything (no blocked beam) into a third apparatus, which is aligned again along the z axis, but now transmits only particles with spin z of $-\frac{\hbar}{2}$. In this case, the middle (x axis) apparatus has no effect since it doesn't filter the particles at all, with the result that we're feeding a stream of $+\frac{\hbar}{2}$ particles into an apparatus that detects only spin $-\frac{\hbar}{2}$. In this case, nothing will get through.

Now let's look at a somewhat more complex situation. We now have a stream of spin-1 particles moving along the y axis into an apparatus with a **B** field aligned on the the z axis. Because m has three possible values $(\pm\hbar,0)$, the output will be split into 3 beams. Suppose we take only the $+\hbar$ beam and feed it into a second apparatus in which the **B** field is rotated by an angle θ relative to the first. What fraction of the particles will get through?

To solve this, we need the unitary rotation operator that relates the two apparatuses. We worked this out before for both spin-$\frac{1}{2}$ and spin-1, and the result we need is the matrix

$$U[R(\theta)] = D^{(1)}[R(\theta)] = I^{(1)} + \frac{\left(\hat{\theta}\cdot\mathbf{J}^{(1)}\right)^2}{\hbar^2}(\cos\theta - 1) - \frac{i\hat{\theta}\cdot\mathbf{J}^{(1)}}{\hbar}\sin\theta \tag{4}$$

Here, $\hat{\theta}$ is a unit vector along the axis of rotation and $\mathbf{J}^{(1)}$ is the angular momentum operator for spin-1. In our case, the rotation is around the y axis, so

$$\hat{\theta} = \hat{\mathbf{y}} = (0,1,0) \tag{5}$$

The components of $\mathbf{J}^{(1)}$ are given in Shankar's equations 12.5.23 and 12.5.24. We need only $J_y^{(1)}$:

$$J_y^{(1)} = \frac{i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{6}$$

We then have

$$\frac{\hat{\theta}\cdot\mathbf{J}^{(1)}}{\hbar} = \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{7}$$

$$\frac{\left(\hat{\theta}\cdot\mathbf{J}^{(1)}\right)^2}{\hbar^2} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{8}$$

From 4 we have

$$U[R(\theta)] = \begin{bmatrix} 1+\frac{\cos\theta-1}{2} & \frac{\sin\theta}{\sqrt{2}} & -\frac{\cos\theta-1}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ -\frac{\cos\theta-1}{2} & -\frac{\sin\theta}{\sqrt{2}} & 1+\frac{\cos\theta-1}{2} \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{bmatrix} \tag{10}$$

The first column is the rotated version of the state $\begin{bmatrix}1\\0\\0\end{bmatrix}$, so a particle in this rotated state has probability of $\left(\frac{1+\cos\theta}{2}\right)^2$ of being in the $+\hbar$ spin state, so this is the fraction of particles leaving the first apparatus that will pass the second. [As a check, note that the sums of the squares of the elements in each column of 10 are 1.]

COMMENTS

Remark 1. I have some confusion about your solution to problem 14.05.03, Principles of Quantum Mechanics, Shankar. (Link: http://physicspages.com/pdf/Shankar/Shankar%20 %2014.05.04.pdf) "Now suppose we return to the case where the first apparatus transmits only spin z of +h/2 but the second (aligned along the x axis) transmits everything (no blocked beam) into a third apparatus, which is aligned again along the z axis, but now transmits only particles with spin z of -h/2. In this case, the middle (x axis) apparatus has no effect since it doesn't filter the particles at all, with the result that we're feeding a stream of +h/2 particles into an apparatus that detects only spin -h/2. In this case, nothing will get through." My opinion is: When the spin +z electrons pass through the second SG (along x axis) apparatus, the outcome is electrons in the +x and -x spin state. The state is no more +z spin. So when the +x and -x spin electrons pass through the 3rd SG apparatus (along z axis) 50% of them should get through.

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I originally thought the same thing, but I think the point is that we aren't allowed to look at the output of the middle detector, so we don't ever measure a particle's x-spin. In that case, all the particles remain in the $+z$ state, so they all get blocked in the third detector. I suspect you're right that if we did measure the x-spin in the middle detector that would place the particle in either the $+x$ or $-x$ state, with z-spin undetermined, so that 50% of them would indeed get through the third detector. Shankar gives zero as the answer at the back of the book, so my guess is that's what's happening.