TOTAL J FOR SUM OF TWO ANGULAR MOMENTA

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.1.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

When we add two spins (or angular momenta) in quantum mechanics, we can express the states in one of two ways. The first is in the vector space which is the direct product of the two spaces for the two spins. This is called the *product space* and formally is

$$\mathbb{V}_p = \mathbb{V}_1 \otimes \mathbb{V}_2 \tag{1}$$

where $\mathbb{V}_i$ is the vector space of the single spin i. If we're interested in the total spin $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, we could also use the total-j vector space, which is the direct sum of the two spin spaces:

$$\mathbb{V}_t = \mathbb{V}_1 \oplus \mathbb{V}_2 \tag{2}$$

As each space is complete, we can express any state in terms of a basis from either space. We've seen an example of this when adding two spin-$\frac{1}{2}$ systems.

In general, if we have two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, we would like to be able to write a state in one space as a linear combination of states from the other space. The Clebsch-Gordan coefficients allow us to this. Calculating the C-G coefficients in general is quite complicated, but for systems with small spins or angular momenta, Shankar gives a method that is simpler than the more tedious brute-force method. We ground through one of these brute-force calculations earlier for the addition of spin-$\frac{1}{2}$ and another, arbitrary, spin.

In this post, we'll work through Shankar's method for the explicit case of adding spin-$\frac{1}{2}$ and spin-1 so you can see how the calculations are done.

We have two sets of kets. In the product space, each ket is labelled by the two spins and their z components, as in

$$\mathbb{V}_p = \{|j_1 m_1; j_2 m_2\rangle\} \tag{3}$$

The curly brackets here represent the set of all kets of form $|j_1m_1;j_2m_2\rangle$ where j_i is the value (in units of $\hbar$) of spin $\mathbf{J}_i$ and m_i is its z component.

In the total-j space $\mathbb{V}_t$, the labels are the total spin j, its z component m and the two component spins j_1 and j_2:

$$\mathbb{V}_t = \{|jmj_1j_2\rangle\} \tag{4}$$

To work out the linear combinations, we start with the state where both j and m are maximum, which occurs when $m_1 = j_1$ and $m_2 = j_2$, which gives $j = j_1 + j_2$ and $m = j_1 + j_2$. There is only one member of the set 3 satisfying this condition, so we begin by stating that

$$|(j_1+j_2)(j_1+j_2)j_1j_2\rangle = |j_1j_1;j_2j_2\rangle \tag{5}$$

To get states with lower values of m but the same value of j, we can apply the lowering operator J_- to the LHS of 5 and its equivalent in the product space, which is $J_{1-}+J_{2-}$, to the RHS. We use the formula

$$J_\pm|jmj_1j_2\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j(m-1)j_1j_2\rangle \tag{6}$$

Shankar gives the details of this calculation in the general case; here we'll apply it to $j_1 = 1$ and $j_2 = \frac{1}{2}$. We begin with the top state, where $j = 1+\frac{1}{2} = \frac{3}{2}$ and $m = \frac{3}{2}$:

$$\left|\frac{3}{2}\frac{3}{2}1\frac{1}{2}\right\rangle_t = \left|11;\frac{1}{2}\frac{1}{2}\right\rangle_p \tag{7}$$

In what follows, to simplify the notation, we'll omit j_1j_2 from the total-j kets (since they are always $1\frac{1}{2}$) and also omit j_1 and j_2 from the product kets (again, because they are always 1 and $\frac{1}{2}$). We'll use a subscript t for a total-j ket and p for a product space ket. In this notation 7 is

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle_t = \left|1\frac{1}{2}\right\rangle_p \tag{8}$$

Now we apply the lowering operator to both sides. On the LHS, we have

$$J_-\left|\frac{3}{2}\frac{3}{2}\right\rangle_t = \hbar\sqrt{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)}\left|\frac{3}{2}\frac{1}{2}\right\rangle_t \tag{9}$$

$$= \sqrt{3}\hbar\left|\frac{3}{2}\frac{1}{2}\right\rangle_t \tag{10}$$

On the RHS, we have (remember that J_{1-} operates only on spin 1 and J_{2-} only on spin 2):

$$\left(J_{1-}+J_{2-}\right)\left|1\frac{1}{2}\right\rangle_{p} = \hbar\sqrt{(1+1)(1-1+1)}\left|0\frac{1}{2}\right\rangle_{p}+ \tag{11}$$

$$\hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|1,-\frac{1}{2}\right\rangle_{p} \tag{12}$$

$$= \sqrt{2}\hbar\left|0\frac{1}{2}\right\rangle_{p}+\hbar\left|1,-\frac{1}{2}\right\rangle_{p} \tag{13}$$

Combining 10 and 13, we find

$$\left|\frac{3}{2}\frac{1}{2}\right\rangle_{t} = \sqrt{\frac{2}{3}}\left|0\frac{1}{2}\right\rangle_{p}+\frac{1}{\sqrt{3}}\left|1,-\frac{1}{2}\right\rangle_{p} \tag{14}$$

To get the next lower value of m, we apply lowering operators again:

$$J_{-}\left|\frac{3}{2}\frac{1}{2}\right\rangle_{t} = 2\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle_{t} \tag{15}$$

$$\left(J_{1-}+J_{2-}\right)\left(\sqrt{\frac{2}{3}}\left|0\frac{1}{2}\right\rangle_{p}+\frac{1}{\sqrt{3}}\left|1,-\frac{1}{2}\right\rangle_{p}\right) = \sqrt{\frac{2}{3}}\sqrt{2}\hbar\left|-1,\frac{1}{2}\right\rangle_{p}+ \tag{16}$$

$$\sqrt{\frac{2}{3}}\hbar\left|0,-\frac{1}{2}\right\rangle_{p}+\frac{1}{\sqrt{3}}\sqrt{2}\hbar\left|0,-\frac{1}{2}\right\rangle_{p} \tag{17}$$

$$= \frac{2}{\sqrt{3}}\hbar\left|-1,\frac{1}{2}\right\rangle_{p}+2\sqrt{\frac{2}{3}}\hbar\left|0,-\frac{1}{2}\right\rangle_{p} \tag{18}$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle_{t} = \frac{1}{\sqrt{3}}\hbar\left|-1,\frac{1}{2}\right\rangle_{p}+\sqrt{\frac{2}{3}}\hbar\left|0,-\frac{1}{2}\right\rangle_{p} \tag{19}$$

To get the bottom ket, we could apply the lowering operator again, but it's easier to notice that there is only one way of getting the state $\left|\frac{3}{2},-\frac{3}{2}\right\rangle_{t}$ so we have

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle_{t} = \left|-1,-\frac{1}{2}\right\rangle_{p} \tag{20}$$

This completes the states with $j=\frac{3}{2}$. There are two total-j states with $j=\frac{1}{2}$: one with $m=+\frac{1}{2}$ and the other with $m=-\frac{1}{2}$. To get the state $\left|\frac{1}{2}\frac{1}{2}\right\rangle_{t}$, we observe that it must be a combination of the product kets $\left|1,-\frac{1}{2}\right\rangle_p$ and $\left|0,\frac{1}{2}\right\rangle_p$. These are the same two kets that were combined to get $\left|\frac{3}{2},\frac{1}{2}\right\rangle_t$ in 14. As usual, we're looking for a mutually orthonormal sets of states, so $\left|\frac{1}{2}\frac{1}{2}\right\rangle_t$ must be orthogonal to $\left|\frac{3}{2}\frac{1}{2}\right\rangle_t$ and also be normalized. By inspection, we see that the state must be

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle_t=\sqrt{\frac{2}{3}}\left|1,-\frac{1}{2}\right\rangle_p-\frac{1}{\sqrt{3}}\left|0\frac{1}{2}\right\rangle_p \tag{21}$$

[Actually, we could multiply this by any phase factor $e^{i\alpha}$ for real α, but by convention, the coefficients are taken to be real. A further convention makes the coefficient of the product ket with $m_1=j_1$ positive.]

To get the state $\left|\frac{1}{2},-\frac{1}{2}\right\rangle_t$ we again use lowering operators:

$$J_-\left|\frac{1}{2}\frac{1}{2}\right\rangle_t=\hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle_t \tag{22}$$

$$(J_{1-}+J_{2-})\left(\sqrt{\frac{2}{3}}\left|1,-\frac{1}{2}\right\rangle_p-\frac{1}{\sqrt{3}}\left|0\frac{1}{2}\right\rangle_p\right)=\sqrt{\frac{2}{3}}\sqrt{2}\hbar\left|0,-\frac{1}{2}\right\rangle_p- \tag{23}$$

$$\frac{\sqrt{2}}{\sqrt{3}}\hbar\left|-1,\frac{1}{2}\right\rangle_p-\frac{1}{\sqrt{3}}\hbar\left|0,-\frac{1}{2}\right\rangle_p \tag{24}$$

$$=\frac{\hbar}{\sqrt{3}}\left|0,-\frac{1}{2}\right\rangle_p-\sqrt{\frac{2}{3}}\hbar\left|-1,\frac{1}{2}\right\rangle_p \tag{25}$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_t=\frac{1}{\sqrt{3}}\left|0,-\frac{1}{2}\right\rangle_p-\sqrt{\frac{2}{3}}\left|-1,\frac{1}{2}\right\rangle_p \tag{26}$$

This completes the transformations.

From here, it's actually not too hard to construct the matrix J^2 in the product basis. We first note that J^2 in the total-j basis is diagonal, with the diagonal entries being the eigenvalues, which are the values of $j(j+1)$ for the 6 states. If we list the states in the order

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle_t,\left|\frac{3}{2}\frac{1}{2}\right\rangle_t,\left|\frac{1}{2}\frac{1}{2}\right\rangle_t,\left|\frac{1}{2}-\frac{1}{2}\right\rangle_t,\left|\frac{3}{2},-\frac{1}{2}\right\rangle_t,\left|\frac{3}{2},-\frac{3}{2}\right\rangle_t \tag{27}$$

then the eigenvalues are $\frac{15}{4}\hbar^2$, $\frac{15}{4}\hbar^2$, $\frac{3}{4}\hbar^2$, $\frac{3}{4}\hbar^2$, $\frac{15}{4}\hbar^2$, $\frac{15}{4}\hbar^2$ so we have

$$J_t^2 = \frac{3\hbar^2}{4}\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \tag{28}$$

To construct J_p^2, we observe that the kets 27 are the eigenvectors of J^2 (in both bases) and we know that the unitary matrix U whose columns are the normalized eigenvectors of J_p^2 will diagonalize J_p^2. In this case, we already have the diagonalized form, which is just J_t^2, so we know that

$$U^T J_p^2 U = J_t^2 \tag{29}$$

Since U is unitary, $U^T = U^{-1}$, so we get

$$J_p^2 = U J_t^2 U^T \tag{30}$$

Using the eigenvector order given in 27 to order the columns of U, we have

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

We can now just do the matrix multiplications (I used Maple, since multiplying 6×6 matrices is quite tedious), and we find

$$J_p^2 = \hbar^2 \begin{bmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4} & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4} & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{bmatrix} \tag{32}$$

To finish, we return to the general results given by Shankar. First, for the general state $|j_1 j_1; j_2 j_2\rangle_p$ we can find the total angular momentum by operating with

$$J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+} \tag{33}$$

[This formula was derived earlier.] Since the state $|j_1j_1;j_2j_2\rangle_p$ has maximum values for m_1 and m_2, operating with J_{1+} or J_{2+} will give zero. Therefore

$$J^2|j_1j_1;j_2j_2\rangle_p = \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z}\right)|j_1j_1;j_2j_2\rangle_p \tag{34}$$
$$= [j_1(j_1+1) + j_2(j_2+1) + 2m_1m_2]\hbar^2|j_1j_1;j_2j_2\rangle_p \tag{35}$$
$$= [j_1(j_1+1) + j_2(j_2+1) + 2j_1j_2]\hbar^2|j_1j_1;j_2j_2\rangle_p \tag{36}$$
$$= [(j_1+j_2)(j_1+j_2+1)]\hbar^2|j_1j_1;j_2j_2\rangle_p \tag{37}$$

Thus the total j value is $j = j_1 + j_2$.

The second exercise is a bit messier, since we're dealing with the top ket whose j value is one unit less than the maximum, which is given by Shankar's equation 15.2.8.

$$|j_1+j_2-1, j_1+j_2-1\rangle_t = \frac{1}{\sqrt{j_1+j_2}}\left[\sqrt{j_1}|j_1,j_2-1\rangle_p - \sqrt{j_2}|j_1-1,j_2\rangle_p\right] \tag{38}$$

This time, operating with 33 must include the two terms with raising operators, so we need to use 6. We'll deal with these terms first. We note that operating with J_{1+} on the first term in 38 gives zero, since $m_1 = j_1$, and similarly for J_{2+}on the second term. We're left with

$$J_{1+}J_{2-}|j_1-1,j_2\rangle_p = \sqrt{2j_2}\hbar J_{1+}|j_1-1,j_2-1\rangle_p \tag{39}$$
$$= \sqrt{2j_2}\hbar^2\sqrt{2j_1}|j_1,j_2-1\rangle_p \tag{40}$$
$$= 2\sqrt{j_1j_2}\hbar^2|j_1,j_2-1\rangle_p \tag{41}$$
$$J_{1-}J_{2+}|j_1,j_2-1\rangle_p = 2\sqrt{j_1j_2}\hbar^2|j_1-1,j_2\rangle_p \tag{42}$$

Combining these two results in 38 we have, for these terms

$$(J_{1+}J_{2-} + J_{1-}J_{2+})\left[\sqrt{j_1}|j_1,j_2-1\rangle_p - \sqrt{j_2}|j_1-1,j_2\rangle_p\right] = \tag{43}$$
$$2j_1\sqrt{j_2}\hbar^2|j_1-1,j_2\rangle_p - 2j_2\sqrt{j_1}\hbar^2|j_1,j_2-1\rangle_p \tag{44}$$

Now for the first 3 terms in 33. First, we apply them to the first term in 38:

$$\left(J_1^2 + J_2^2 + 2J_{1z}J_{2z}\right)\sqrt{j_1}|j_1,j_2-1\rangle_p = \sqrt{j_1}\hbar^2[j_1(j_1+1) + j_2(j_2+1) + 2j_1(j_2-1)]|j_1,j_2-1\rangle_p \tag{45}$$

Combining this with 44 we get the coefficient of $|j_1, j_2 - 1\rangle_p$ to be

$$\sqrt{j_1}\hbar^2\left[j_1(j_1+1)+j_2(j_2+1)+2j_1(j_2-1)-2j_2\right] = \quad (46)$$

$$\sqrt{j_1}\hbar^2\left[(j_1+j_2-1)(j_1+j_2)\right] \quad (47)$$

Now we apply $\left(J_1^2+J_2^2+2J_{1z}J_{2z}\right)$ to the second term in 38:

$$-\left(J_1^2+J_2^2+2J_{1z}J_{2z}\right)\sqrt{j_2}\,|j_1-1,j_2\rangle_p = -\sqrt{j_2}\hbar^2\left[j_1(j_1+1)+j_2(j_2+1)+2(j_1-1)j_2\right]|j_1-1,j_2\rangle \quad (48)$$

Again, combining this with 44 we get the coefficient of $|j_1-1, j_2\rangle_p$ to be

$$-\sqrt{j_2}\hbar^2\left[j_1(j_1+1)+j_2(j_2+1)+2(j_1-1)j_2-2j_1\right] = \quad (49)$$

$$-\sqrt{j_2}\hbar^2\left[(j_1+j_2-1)(j_1+j_2)\right] \quad (50)$$

Thus the net result of operating on 38 with J^2 is to multiply by $(j_1+j_2-1)(j_1+j_2)$, this state has angular momentum j_1+j_2-1.

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