CLEBSCH-GORDAN COEFFICIENTS - EXAMPLES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercises 15.2.2 - 15.2.3.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen a detailed example of calculating Clebsch-Gordan coefficients by using the angular momentum lowering operator, where we calculated the coefficients for the case of spin-1 combined with spin-$\frac{1}{2}$. Here we'll give a slightly more involved example by combining two spin-1 systems.

Shankar gives the conditions satisfied by the CG coefficients, so we'll apply these here. We start with the combined system with the maximum values of j (the total angular momentum number) and m (the z component of the total angular momentum), which here means we have $j=m=2$. As there is only one member of the product space ($m_1=m_2=1$) satisfying this condition, we have

$$\left|22\right\rangle_t=\left|11\right\rangle_p \tag{1}$$

As in the previous post, we give the ket in the total-j space as $\left|jm\right\rangle_t$ (subscript t for 'total-j') and in the product space as $\left|m_1m_2\right\rangle_p$ (subscript p for 'product'). The two individual spins j_1 and j_2 are always the same in all cases, so we omit them from the notation.

We now apply the lowering operator to both sides to generate the next state. This operator is, in the total-j space:

$$J_-\left|jm\right\rangle_t=\hbar\sqrt{(j+m)(j-m+1)}\left|j(m-1)\right\rangle_t \tag{2}$$

In the product space, we have

$$(J_{1-}+J_{2-})\left|m_1m_2\right\rangle_p=\hbar\sqrt{(j_1+m_1)(j_1-m_1+1)}\left|(m_1-1)m_2\right\rangle_p+ \tag{3}$$

$$\hbar\sqrt{(j_2+m_2)(j_2-m_2+1)}\left|m_1(m_2-1)\right\rangle_p \tag{4}$$

In what follows, we'll omit the $\hbar$ since it always occurs in every term on both sides of the equation, so it always cancels out in the final result.

Starting from 1 we have

$$J_-\left|22\right\rangle_t = 2\left|21\right\rangle_t \tag{5}$$

$$(J_{1-}+J_{2-})\left|11\right\rangle_p = \sqrt{2}\left|01\right\rangle_p+\sqrt{2}\left|10\right\rangle_p \tag{6}$$

$$\left|21\right\rangle_t = \frac{1}{\sqrt{2}}\left|01\right\rangle_p+\frac{1}{\sqrt{2}}\left|10\right\rangle_p \tag{7}$$

For the next state, we have

$$J_-\left|21\right\rangle_t = \sqrt{6}\left|20\right\rangle_t \tag{8}$$

$$\frac{1}{\sqrt{2}}(J_{1-}+J_{2-})\left(\left|01\right\rangle_p+\left|10\right\rangle_p\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\left|-11\right\rangle_p+\sqrt{2}\left|00\right\rangle_p\right)+ \tag{9}$$

$$\frac{1}{\sqrt{2}}\left(\sqrt{2}\left|00\right\rangle_p+\sqrt{2}\left|1,-1\right\rangle_p\right) \tag{10}$$

$$= \left|-11\right\rangle_p+2\left|00\right\rangle_p+\left|1,-1\right\rangle_p \tag{11}$$

$$\left|20\right\rangle_t = \frac{1}{\sqrt{6}}\left|-11\right\rangle_p+\sqrt{\frac{2}{3}}\left|00\right\rangle_p+\frac{1}{\sqrt{6}}\left|1,-1\right\rangle_p \tag{12}$$

[The last line equates the first line with the fourth line.]
To get the states with negative m, we can use equation 15.2.11 in Shankar:

$$\left\langle m_1m_2\left|jm\right.\right\rangle = (-1)^{j_1+j_2-j}\left\langle -m_1,-m_2\left|j,-m\right.\right\rangle \tag{13}$$

Here, the bracket $\left\langle m_1m_2\left|jm\right.\right\rangle$ is the CG coefficient that multiplies $\left|m_1m_2\right\rangle$ in the expansion of $\left|jm\right\rangle$. For example, in 12

$$\left\langle -11\left|20\right.\right\rangle = \frac{1}{\sqrt{6}} \tag{14}$$

$$\left\langle 00\left|20\right.\right\rangle = \sqrt{\frac{2}{3}} \tag{15}$$

$$\left\langle 1,-1\left|20\right.\right\rangle = \frac{1}{\sqrt{6}} \tag{16}$$

Using 13 and 7, we have $j_1+j_2-j=1+1-2=0$, so

$$\left|2,-1\right\rangle_t = \frac{1}{\sqrt{2}}\left|0,-1\right\rangle_p+\frac{1}{\sqrt{2}}\left|-10\right\rangle_p \tag{17}$$

The final ket in the column with $j=2$ is

$$\left|2,-2\right\rangle_t = \left|-1,-1\right\rangle_p \tag{18}$$

For the next column, we have $j=1$ and the top entry therefore has $m=1$. This total-j state $|11\rangle_t$ must be a combination of the product states $|10\rangle_p$ and $|01\rangle_p$, must be orthogonal to 7 and the coefficient of the term with $m_1=j_1$ is by convention positive. By inspection, we have

$$|11\rangle_t = -\frac{1}{\sqrt{2}}|01\rangle_p + \frac{1}{\sqrt{2}}|10\rangle_p \tag{19}$$

Note that $m_1=j_1=1$ in the second term, so it's the positive one.

To get the next ket, we apply the lowering operator again:

$$J_-|11\rangle_t = \sqrt{2}|10\rangle_t \tag{20}$$

$$\frac{1}{\sqrt{2}}(J_{1-}+J_{2-})\left(|01\rangle_p - |10\rangle_p\right) = -\frac{1}{\sqrt{2}}\left(\sqrt{2}|-11\rangle_p + \sqrt{2}|00\rangle_p\right) + \tag{21}$$

$$\frac{1}{\sqrt{2}}\left(\sqrt{2}|00\rangle_p + \sqrt{2}|1,-1\rangle_p\right) \tag{22}$$

$$= -|-11\rangle_p + |1,-1\rangle_p \tag{23}$$

$$|10\rangle_t = -\frac{1}{\sqrt{2}}|-11\rangle_p + \frac{1}{\sqrt{2}}|1,-1\rangle_p \tag{24}$$

We apply 13 to get the final entry in this column. This time $j_1+j_2-j=1$ so

$$|1,-1\rangle_t = \frac{1}{\sqrt{2}}|0,-1\rangle_p - \frac{1}{\sqrt{2}}|-10\rangle_p \tag{25}$$

Finally, there is one total-j ket in the third column, where $j=m=0$. This time, the ket $|00\rangle_t$ must be a combination of $|-11\rangle_p$, $|00\rangle_p$ and $|1,-1\rangle_p$, and must be orthogonal to both 12 and 24. Suppose

$$|00\rangle_t = a|-11\rangle_p + b|00\rangle_p + c|1,-1\rangle_p \tag{26}$$

Then the orthogonality conditions tell us that

$$\frac{a}{\sqrt{6}} + \sqrt{\frac{2}{3}}b + \frac{c}{\sqrt{6}} = 0 \tag{27}$$

$$-\frac{a}{\sqrt{2}} + \frac{c}{\sqrt{2}} = 0 \tag{28}$$

We can solve these to find

$$c = a \tag{29}$$
$$2a+2b = 0 \tag{30}$$
$$b = -a \tag{31}$$

We can find a from the normalization condition

$$a^2+b^2+c^2 = 1 \tag{32}$$
$$3a^2 = 1 \tag{33}$$

Thus we have

$$a = c = \frac{1}{\sqrt{3}} \tag{34}$$
$$b = -\frac{1}{\sqrt{3}} \tag{35}$$
$$|00\rangle_t = \frac{1}{\sqrt{3}}|-11\rangle_p - \frac{1}{\sqrt{3}}|00\rangle_p + \frac{1}{\sqrt{3}}|1,-1\rangle_p \tag{36}$$

These CG coefficients agree with those given in Griffiths's Table 4.8, for example.

A final comment on the dimensionality of the various spaces. If we combine two single spins j_1 and j_2, then the dimensionality of the product space $j_1 \otimes j_2$ is $(2j_1+1)(2j_2+1)$, since there are $2j_i+1$ possible m_i values for spin j_i. In the above example, both j_1 and j_2 are 1, so the dimensionality of $1 \otimes 1$ is $3 \times 3 = 9$. The dimensionality of the corresponding total-j space is the sum of the dimensions for each possible value of j within this space. For $1 \otimes 1$, there are 5 states with $j = 2$, 3 states with $j = 1$ and 1 state with $j = 0$, for a total of $5+3+1 = 9$.

If we combine more than 2 spins, we can apply the same argument, provided we count the number of total-j states properly. For $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$, the product space contains $2 \times 2 \times 2 = 8$ dimensions, so the total-j space must also contain 8 dimensions. In the total-j space, j can be $\frac{3}{2}$ or $\frac{1}{2}$. We have 4 states with $j = \frac{3}{2}$. For $j = \frac{1}{2}$, we can have $m = \pm\frac{1}{2}$. Consider the ket $\left|\frac{1}{2}\frac{1}{2}\right\rangle_t$. It must be a combination of product states where two spins are up and one is down (that is, two of the m_i are $+\frac{1}{2}$ and one is $-\frac{1}{2}$), so we have

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle_t = a|\uparrow\uparrow\downarrow\rangle + b|\uparrow\downarrow\uparrow\rangle + c|\downarrow\uparrow\uparrow\rangle \tag{37}$$

The only constraints we have on a, b and c are (1) the state $\left|\frac{1}{2}\frac{1}{2}\right\rangle_t$ must be orthogonal to $\left|\frac{3}{2}\frac{1}{2}\right\rangle_t$ and (2) the state must be normalized. As we have only 2 constraints on 3 unknowns, the subspace occupied by $\left|\frac{1}{2}\frac{1}{2}\right\rangle_t$ is two-dimensional. The same argument applies to $\left|\frac{1}{2},-\frac{1}{2}\right\rangle_t$ so it, too, is two-dimensional. Thus the total dimensionality of the total-j space is $4+2+2=8$, or in other words, $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$.