

CLEBSCH-GORDAN COEFFICIENTS FOR ADDITION OF SPIN-1/2 AND GENERAL L

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've looked at the problem of adding spin- $\frac{1}{2}$ to another arbitrary spin s_2 before, but Shankar provides a different method to get this result. In Shankar's book, the problem is to add a general angular momentum \mathbf{L} to a spin- $\frac{1}{2}$ system. In the product space, there are four states in such a system. In all of these states ℓ and $s = \frac{1}{2}$ are always the same. We wish to construct the total- j state for a given total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and a specified z component of total angular momentum m from these four states. In a given product state, the z component of spin is either $m_s = \pm\frac{1}{2}$ and since we must have $m = m_\ell + m_s$, the orbital z component must be $m_\ell = m - m_s$. Therefore, for a given m , the two possible total- j states are

$$(0.1) \quad \left| \ell + \frac{1}{2}, m \right\rangle_t = \alpha \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

$$(0.2) \quad \left| \ell - \frac{1}{2}, m \right\rangle_t = \alpha' \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta' \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

As usual, a subscript t on a ket indicates a total- j state and a subscript p indicates a product state. We've omitted ℓ and s in the product states since they are always the same.

The coefficients α , β , α' and β' can be determined by applying several constraints. The two states must be orthonormal which gives us three constraints

$$(0.3) \quad \alpha^2 + \beta^2 = 1$$

$$(0.4) \quad \alpha'^2 + \beta'^2 = 1$$

$$(0.5) \quad \alpha\alpha' + \beta\beta' = 0$$

[As usual for Clebsch-Gordan coefficients, we're taking them to be real, so we don't need to indicate norms in these equations.] To get a fourth constraint, we can apply the total angular momentum operator J^2 in the form

$$\begin{aligned}
 (0.6) \quad J^2 &= (\mathbf{L} + \mathbf{S})^2 \\
 (0.7) \quad &= L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S} \\
 (0.8) \quad &= L^2 + S^2 + 2L_z S_z + L_- S_+ + L_+ S_-
 \end{aligned}$$

Applying this operator to the LHS of 0.1 and 0.2, we have

$$\begin{aligned}
 (0.9) \quad J^2 \left| \ell + \frac{1}{2}, m \right\rangle_t &= \hbar^2 \left(\ell + \frac{1}{2} \right) \left(\ell + \frac{3}{2} \right) \left| \ell + \frac{1}{2}, m \right\rangle_t \\
 (0.10) \quad J^2 \left| \ell - \frac{1}{2}, m \right\rangle_t &= \hbar^2 \left(\ell - \frac{1}{2} \right) \left(\ell + \frac{1}{2} \right) \left| \ell - \frac{1}{2}, m \right\rangle_t
 \end{aligned}$$

On the RHS, we need the formulas for the raising and lowering operators (which also apply if we replace J by L or S):

$$(0.11) \quad J_{\pm} |jm\rangle_p = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j(m \pm 1)\rangle_p$$

To apply this formula to 0.1, for example, we see that $L_- S_+ \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p = 0$ since m_s is at its maximum value so applying the raising operator to that state gives zero. To calculate $L_- S_+ \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$ we need to consider the two operators in turn.

We have

(0.12)

$$S_+ \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p = \hbar \sqrt{(s - m_s)(s + m_s + 1)} \left| m + \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

$$(0.13) \quad = \hbar \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right) \left(\frac{1}{2} - \frac{1}{2} + 1\right)} \left| m + \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

$$(0.14) \quad = \hbar \left| m + \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

(0.15)

$$L_- \hbar \left| m + \frac{1}{2}, \frac{1}{2} \right\rangle_p = \hbar^2 \sqrt{(\ell + m_\ell)(\ell - m_\ell + 1)} \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

$$(0.16) \quad = \hbar^2 \sqrt{\left(\ell + m + \frac{1}{2}\right) \left(\ell - m - \frac{1}{2} + 1\right)} \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

$$(0.17) \quad = \hbar^2 \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p$$

 Similarly we have $L_+ S_- \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p = 0$ and

(0.18)

$$S_- \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p = \hbar \sqrt{(s + m_s)(s - m_s + 1)} \left| m - \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

$$(0.19) \quad = \hbar \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right) \left(\frac{1}{2} - \frac{1}{2} + 1\right)} \left| m - \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

$$(0.20) \quad = \hbar \left| m - \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

(0.21)

$$L_+ \hbar \left| m - \frac{1}{2}, -\frac{1}{2} \right\rangle_p = \hbar^2 \sqrt{(\ell - m_\ell)(\ell + m_\ell + 1)} \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

$$(0.22) \quad = \hbar^2 \sqrt{\left(\ell - m + \frac{1}{2}\right) \left(\ell + m - \frac{1}{2} + 1\right)} \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

$$(0.23) \quad = \hbar^2 \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

Therefore (I'll drop the factor of \hbar^2 from now on, since it cancels out in the end):

(0.24)

$$[L^2 + S^2 + 2L_z S_z + L_- S_+ + L_+ S_-] \left[\alpha \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p \right] =$$

$$(0.25) \quad \left[\ell(\ell+1) + \frac{3}{4} \right] \left[\alpha \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p \right] +$$

$$(0.26) \quad \left[\alpha \left(m - \frac{1}{2} \right) \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p - \beta \left(m + \frac{1}{2} \right) \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p \right] +$$

$$(0.27) \quad \sqrt{\left(\ell + \frac{1}{2} \right)^2 - m^2} \left[\alpha \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p \right] =$$

(0.28)

$$\alpha \left[\ell(\ell+1) + \frac{3}{4} + \left(m - \frac{1}{2} \right) + \sqrt{\left(\ell + \frac{1}{2} \right)^2 - m^2} \right] \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle +$$

(0.29)

$$\beta \left[\ell(\ell+1) + \frac{3}{4} - \left(m + \frac{1}{2} \right) + \sqrt{\left(\ell + \frac{1}{2} \right)^2 - m^2} \right] \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p =$$

$$(0.30) \quad \left(\ell + \frac{1}{2} \right) \left(\ell + \frac{3}{2} \right) \left[\alpha \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \beta \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p \right]$$

where the last equality comes from equating the result with 0.9.

To get the ratio β/α from here is just algebra, so here are the steps. We start by noting that

(0.31)

$$\ell(\ell+1) + \frac{3}{4} + \left(m - \frac{1}{2}\right) + \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} = \left(\ell + \frac{1}{2}\right)^2 + m + \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2}$$

(0.32)

$$\ell(\ell+1) + \frac{3}{4} - \left(m + \frac{1}{2}\right) + \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} = \left(\ell + \frac{1}{2}\right)^2 - m + \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2}$$

(0.33)

$$\left(\ell + \frac{1}{2}\right) \left(\ell + \frac{3}{2}\right) = \left(\ell + \frac{1}{2}\right)^2 + \left(\ell + \frac{1}{2}\right)$$

Dividing 0.30 through by α , collecting terms and cancelling the $(\ell + \frac{1}{2})^2$ terms we get

(0.34)

$$\left[\sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} - \left(\ell + \frac{1}{2} + m\right) \right] \frac{\beta}{\alpha} = \left(\ell + \frac{1}{2} - m\right) - \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2}$$

Finally, we note that

$$(0.35) \quad \sqrt{\left(\ell + \frac{1}{2}\right)^2 - m^2} = \sqrt{\left(\ell + \frac{1}{2} + m\right) \left(\ell + \frac{1}{2} - m\right)}$$

Therefore

$$(0.36) \quad \frac{\beta}{\alpha} = \frac{\left(\ell + \frac{1}{2} - m\right) - \sqrt{\left(\ell + \frac{1}{2} + m\right) \left(\ell + \frac{1}{2} - m\right)}}{\sqrt{\left(\ell + \frac{1}{2} - m\right) \left(\ell + \frac{1}{2} - m\right) - \left(\ell + \frac{1}{2} + m\right)}}$$

$$(0.37) \quad = \sqrt{\frac{\ell + \frac{1}{2} - m}{\ell + \frac{1}{2} + m}} \left[\frac{-\sqrt{\left(\ell + \frac{1}{2} + m\right)} + \sqrt{\left(\ell + \frac{1}{2} - m\right)}}{\sqrt{\left(\ell + \frac{1}{2} - m\right)} - \sqrt{\left(\ell + \frac{1}{2} + m\right)}} \right]$$

$$(0.38) \quad = \sqrt{\frac{\ell + \frac{1}{2} - m}{\ell + \frac{1}{2} + m}}$$

From here, we can use 0.3 to get

$$(0.39) \quad \alpha^2 + \beta^2 = \alpha^2 \left(1 + \frac{\beta^2}{\alpha^2} \right)$$

$$(0.40) \quad = \alpha^2 \left(1 + \frac{\ell + \frac{1}{2} - m}{\ell + \frac{1}{2} + m} \right)$$

$$(0.41) \quad = \alpha^2 \frac{2\ell + 1}{\ell + \frac{1}{2} + m} = 1$$

$$(0.42) \quad \alpha = \sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$$

$$(0.43) \quad \beta = \alpha \sqrt{\frac{\ell + \frac{1}{2} - m}{\ell + \frac{1}{2} + m}}$$

$$(0.44) \quad = \sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$$

We can get α' and β' from 0.4 and 0.5:

$$(0.45) \quad \frac{\beta'}{\alpha'} = -\frac{\alpha}{\beta} = -\sqrt{\frac{\ell + \frac{1}{2} + m}{\ell + \frac{1}{2} - m}}$$

A bit of algebra gives

$$(0.46) \quad \alpha' = -\sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$$

$$(0.47) \quad \beta' = \sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$$

The sign of α' is determined from the convention that the coefficient of the product ket with the highest m is positive. Combining these results gives Shankar's equation 15.2.20:

$$(0.48) \quad \left| \ell \pm \frac{1}{2}, m \right\rangle_t = \pm \sqrt{\frac{\ell + \frac{1}{2} \pm m}{2\ell + 1}} \left| m - \frac{1}{2}, \frac{1}{2} \right\rangle_p + \sqrt{\frac{\ell + \frac{1}{2} \mp m}{2\ell + 1}} \left| m + \frac{1}{2}, -\frac{1}{2} \right\rangle_p$$

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