CLEBSCH-GORDAN COEFFICIENTS FOR ADDITION OF SPIN-1/2 AND GENERAL L

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've looked at the problem of adding spin-$\frac{1}{2}$ to another arbitrary spin s_2 before, but Shankar provides a different method to get this result. In Shankar's book, the problem is to add a general angular momentum $\mathbf{L}$ to a spin-$\frac{1}{2}$ system. In the product space, there are four states in such a system. In all of these states ℓ and $s=\frac{1}{2}$ are always the same. We wish to construct the total-j state for a given total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$ and a specified z component of total angular momentum m from these four states. In a given product state, the z component of spin is either $m_s=\pm\frac{1}{2}$ and since we must have $m=m_\ell+m_s$, the orbital z component must be $m_\ell=m-m_s$. Therefore, for a given m, the two possible total-j states are

$$\left|\ell+\frac{1}{2},m\right\rangle_t = \alpha\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\beta\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{1}$$

$$\left|\ell-\frac{1}{2},m\right\rangle_t = \alpha'\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\beta'\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{2}$$

As usual, a subscript t on a ket indicates a total-j state and a subscript p indicates a product state. We've omitted ℓ and s in the product states since they are always the same.

The coefficients α, β, α' and β' can be determined by applying several constraints. The two states must be orthonormal which gives us three constraints

$$\begin{aligned}\alpha^2+\beta^2 &= 1 \qquad (3)\\ \alpha'^2+\beta'^2 &= 1 \qquad (4)\\ \alpha\alpha'+\beta\beta' &= 0 \qquad (5)\end{aligned}$$

[As usual for Clebsch-Gordan coefficients, we're taking them to be real, so we don't need to indicate norms in these equations.] To get a fourth constraint, we can apply the total angular momentum operator J^2 in the form

$$\begin{aligned} J^2 &= (\mathbf{L}+\mathbf{S})^2 && (6)\\ &= L^2+S^2+2\mathbf{L}\cdot\mathbf{S} && (7)\\ &= L^2+S^2+2L_zS_z+L_-S_++L_+S_- && (8) \end{aligned}$$

Applying this operator to the LHS of 1 and 2, we have

$$J^2\left|\ell+\frac{1}{2},m\right\rangle_t = \hbar^2\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)\left|\ell+\frac{1}{2},m\right\rangle_t \tag{9}$$

$$J^2\left|\ell-\frac{1}{2},m\right\rangle_t = \hbar^2\left(\ell-\frac{1}{2}\right)\left(\ell+\frac{1}{2}\right)\left|\ell-\frac{1}{2},m\right\rangle_t \tag{10}$$

On the RHS, we need the formulas for the raising and lowering operators (which also apply if we replace J by L or S):

$$J_\pm\left|jm\right\rangle_p = \hbar\sqrt{(j\mp m)(j\pm m+1)}\left|j\,(m-1)\right\rangle_p \tag{11}$$

To apply this formula to 1, for example, we see that $L_-S_+\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p = 0$ since m_s is at its maximum value so applying the raising operator to that state gives zero. To calculate $L_-S_+\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p$ we need to consider the two operators in turn.

We have

$$S_+\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p = \hbar\sqrt{(s-m_s)(s+m_s+1)}\left|m+\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{12}$$

$$= \hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|m+\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{13}$$

$$= \hbar\left|m+\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{14}$$

$$L_-\hbar\left|m+\frac{1}{2},\frac{1}{2}\right\rangle_p = \hbar^2\sqrt{(\ell+m_\ell)(\ell-m_\ell+1)}\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{15}$$

$$= \hbar^2\sqrt{\left(\ell+m+\frac{1}{2}\right)\left(\ell-m-\frac{1}{2}+1\right)}\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{16}$$

$$= \hbar^2\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p \tag{17}$$

Similarly we have $L_+S_-\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p = 0$ and

$$S_-\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p = \hbar\sqrt{(s+m_s)(s-m_s+1)}\left|m-\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{18}$$

$$= \hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|m-\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{19}$$

$$= \hbar\left|m-\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{20}$$

$$L_+\hbar\left|m-\frac{1}{2},-\frac{1}{2}\right\rangle_p = \hbar^2\sqrt{(\ell-m_\ell)(\ell+m_\ell+1)}\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{21}$$

$$= \hbar^2\sqrt{\left(\ell-m+\frac{1}{2}\right)\left(\ell+m-\frac{1}{2}+1\right)}\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{22}$$

$$= \hbar^2\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{23}$$

Therefore (I'll drop the factor of $\hbar^2$ from now on, since it cancels out in the end):

$$\left[L^2+S^2+2L_zS_z+L_-S_++L_+S_-\right]\left[\alpha\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\beta\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right]= \tag{24}$$

$$\left[\ell(\ell+1)+\frac{3}{4}\right]\left[\alpha\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\beta\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right]+ \tag{25}$$

$$\left[\alpha\left(m-\frac{1}{2}\right)\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p-\beta\left(m+\frac{1}{2}\right)\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right]+ \tag{26}$$

$$\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\left[\beta\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\alpha\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right]= \tag{27}$$

$$\alpha\left[\left(\ell(\ell+1)+\frac{3}{4}+\left(m-\frac{1}{2}\right)\right)\left|m-\frac{1}{2},\frac{1}{2}\right\rangle+\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right]+ \tag{28}$$

$$\beta\left[\left(\ell(\ell+1)+\frac{3}{4}-\left(m+\frac{1}{2}\right)\right)\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p+\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p\right]= \tag{29}$$

$$\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)\left[\alpha\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\beta\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p\right] \tag{30}$$

where the last equality comes from equating the result with 9. Equating coefficients in the last equation gives us

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$$\left(\ell(\ell+1)+\frac{3}{4}+\left(m-\frac{1}{2}\right)\right)\alpha+\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\beta=\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)\alpha \tag{31}$$

$$\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}\alpha+\left(\ell(\ell+1)+\frac{3}{4}-\left(m+\frac{1}{2}\right)\right)\beta=\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)\beta \tag{32}$$

Dividing the first equation by α we get

$$\frac{\beta}{\alpha}=\frac{\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)-\left(\ell\left(\ell+1\right)+\frac{3}{4}+\left(m-\frac{1}{2}\right)\right)}{\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}} \tag{33}$$

We can simplify this by noting that

$$\ell\left(\ell+1\right)+\frac{3}{4}+\left(m-\frac{1}{2}\right)=\left(\ell+\frac{1}{2}\right)^2+m \tag{34}$$

$$\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)=\left(\ell+\frac{1}{2}\right)^2+\left(\ell+\frac{1}{2}\right) \tag{35}$$

so that

$$\frac{\beta}{\alpha}=\frac{\left(\ell+\frac{1}{2}\right)-m}{\sqrt{\left(\ell+\frac{1}{2}\right)^2-m^2}} \tag{36}$$

$$=\frac{\ell+\frac{1}{2}-m}{\sqrt{\ell+\frac{1}{2}+m}\sqrt{\ell+\frac{1}{2}-m}} \tag{37}$$

$$=\sqrt{\frac{\ell+\frac{1}{2}-m}{\ell+\frac{1}{2}+m}} \tag{38}$$

From here, we can use 3 to get

The other equation 32 gives the same result for $\frac{\beta}{\alpha}$.

$$\alpha^2+\beta^2=\alpha^2\left(1+\frac{\beta^2}{\alpha^2}\right) \tag{39}$$

$$=\alpha^2\left(1+\frac{\ell+\frac{1}{2}-m}{\ell+\frac{1}{2}+m}\right) \tag{40}$$

$$=\alpha^2\frac{2\ell+1}{\ell+\frac{1}{2}+m}=1 \tag{41}$$

$$\alpha=\sqrt{\frac{\ell+\frac{1}{2}+m}{2\ell+1}} \tag{42}$$

$$\beta=\alpha\sqrt{\frac{\ell+\frac{1}{2}-m}{\ell+\frac{1}{2}+m}} \tag{43}$$

$$=\sqrt{\frac{\ell+\frac{1}{2}-m}{2\ell+1}} \tag{44}$$

We can get α' and β' from 4 and 5:

$$\frac{\beta'}{\alpha'}=-\frac{\alpha}{\beta}=-\sqrt{\frac{\ell+\frac{1}{2}+m}{\ell+\frac{1}{2}-m}} \tag{45}$$

A bit of algebra gives

$$\alpha' = -\sqrt{\frac{\ell+\frac{1}{2}-m}{2\ell+1}} \tag{46}$$

$$\beta' = \sqrt{\frac{\ell+\frac{1}{2}+m}{2\ell+1}} \tag{47}$$

The sign of α' is determined from the convention that the coefficient of the product ket with the highest m is positive. Combining these results gives Shankar's equation 15.2.20:

$$\left|\ell\pm\frac{1}{2},m\right\rangle_t=\pm\sqrt{\frac{\ell+\frac{1}{2}\pm m}{2\ell+1}}\left|m-\frac{1}{2},\frac{1}{2}\right\rangle_p+\sqrt{\frac{\ell+\frac{1}{2}\mp m}{2\ell+1}}\left|m+\frac{1}{2},-\frac{1}{2}\right\rangle_p \tag{48}$$

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