

## PROJECTION OPERATORS FOR SPIN-1/2 + SPIN-1/2

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.5.

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We've seen projection operators in a formal mathematical sense, but in this post, we'll see a practical example of projection operators in spin space. We look at a system of two spin- $\frac{1}{2}$  particles, with spin operators  $\mathbf{S}_1$  and  $\mathbf{S}_2$  for each of the two particles. Now consider the operators

$$(0.1) \quad \mathbb{P}_1 = \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2$$

$$(0.2) \quad \mathbb{P}_2 = \frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2$$

A projection operator projects an arbitrary vector onto a subspace of the vector space in which that vector resides. The two projection operators here project onto orthogonal subspaces, which means if we project some vector  $V$  first with  $\mathbb{P}_1$  and then with  $\mathbb{P}_2$  (or vice versa), we'll end up with the zero vector. Also, if we project  $V$  twice (or more) with the same projection operator, all projections after the first have no further effect. That is

$$(0.3) \quad \mathbb{P}_i \mathbb{P}_j = \delta_{ij} \mathbb{P}_j$$

To show that this is true for the two projection operators above, we can make use of an identity derived earlier:

$$(0.4) \quad (\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = (\mathbf{A} \cdot \mathbf{B})I + i(\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma}$$

which is valid if  $\mathbf{A}$  and  $\mathbf{B}$  commute with  $\boldsymbol{\sigma}$ .

Here  $\mathbf{A}$  and  $\mathbf{B}$  are vector operators that commute with the Pauli matrices  $\boldsymbol{\sigma}$ .

First, we'll look at  $\mathbb{P}_1 \mathbb{P}_2$ :

$$(0.5) \quad \mathbb{P}_1\mathbb{P}_2 = \left[ \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \right] \left[ \frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \right]$$

$$(0.6) \quad = \left[ \frac{3}{4}I + \frac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right] \left[ \frac{1}{4}I - \frac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right]$$

$$(0.7) \quad = \frac{3}{16}I - \frac{2}{16}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{1}{16}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2$$

We can write the last term as

$$(0.8) \quad (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$$

We see that this has the same form as 0.4 with  $\mathbf{A} = \mathbf{B} = \boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_2$ . Since  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$  refer to different spins, they commute, so the identity is valid. We get

$$(0.9) \quad (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 I + i(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_1) \cdot \boldsymbol{\sigma}_2$$

The first term is (using the fact that the square of each Pauli matrix is  $I$ ):

$$(0.10) \quad \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 I = (\sigma_{x1}^2 + \sigma_{y1}^2 + \sigma_{z1}^2) I$$

$$(0.11) \quad = 3I^2$$

$$(0.12) \quad = 3I$$

The cross product is just a shorthand way of writing the commutation relations. To see this, work out the  $x$  component, for example:

$$(0.13) \quad (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_1)_x = \sigma_{y1}\sigma_{z1} - \sigma_{z1}\sigma_{y1} = 2i\sigma_{x1}$$

We can write this as

$$(0.14) \quad (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_1) = i\boldsymbol{\sigma}_1$$

Plugging this into 0.9 we have

$$(0.15) \quad (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = 3I - 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

This gives, from 0.7

$$(0.16) \quad \mathbb{P}_1\mathbb{P}_2 = \frac{3}{16}I - \frac{2}{16}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3}{16}I + \frac{2}{16}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 0$$

A similar calculation shows that

$$(0.17) \quad \mathbb{P}_2\mathbb{P}_1 = 0$$

We can also calculate

$$(0.18) \quad \mathbb{P}_1\mathbb{P}_1 = \left[ \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \right] \left[ \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \right]$$

$$(0.19) \quad = \left[ \frac{3}{4}I + \frac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right] \left[ \frac{3}{4}I + \frac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right]$$

$$(0.20) \quad = \frac{9}{16}I + \frac{6}{16}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \frac{1}{16}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2$$

$$(0.21) \quad = \frac{12}{16}I + \frac{4}{16}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

$$(0.22) \quad = \frac{3}{4}I + \frac{1}{4}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

$$(0.23) \quad = \mathbb{P}_1$$

A similar calculation shows that

$$(0.24) \quad \mathbb{P}_2\mathbb{P}_2 = \mathbb{P}_2$$

To find the subspace to which each projection operator projects, we can use the explicit matrix forms in the product basis for the projections. We have

$$(0.25) \quad \mathbb{P}_1 = \frac{3}{4}I + \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2$$

$$(0.26) \quad = \frac{3}{4}I + \frac{1}{\hbar^2} \left( \frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_{1-}S_{2+} + S_{1z}S_{2z} \right)$$

$$(0.27) \quad = \frac{3}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$(0.28) \quad \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(0.29) \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly

$$(0.30) \quad \mathbb{P}_2 = \frac{1}{4}I - \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2$$

$$(0.31) \quad = \frac{3}{4}I - \frac{1}{\hbar^2} \left( \frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_{1-}S_{2+} + S_{1z}S_{2z} \right)$$

$$(0.32) \quad = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} -$$

$$(0.33) \quad \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(0.34) \quad = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we apply these projections to an arbitrary vector  $V$ , we have

$$(0.35) \quad \mathbb{P}_1 V = \mathbb{P}_1 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$(0.36) \quad = \begin{bmatrix} a \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) \\ d \end{bmatrix}$$

$$(0.37) \quad = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}(b+c) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus  $\mathbb{P}_1$  projects  $V$  into the subspace spanned by the basis vectors of the 3-dimensional spin-1 subspace.

For  $\mathbb{P}_2$  we have

$$(0.38) \quad \mathbb{P}_2 V = \mathbb{P}_2 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$(0.39) \quad = \begin{bmatrix} 0 \\ \frac{1}{2}(b-c) \\ \frac{1}{2}(-b+c) \\ 0 \end{bmatrix}$$

$$(0.40) \quad = \frac{1}{\sqrt{2}}(b-c) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Thus  $\mathbb{P}_2$  projects onto the 1-dimensional spin-0 subspace.

In the total- $j$  basis

$$(0.41) \quad S^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$$

$$(0.42) \quad \mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$$

In both the spin-1 and spin-0 states, the eigenvalues of  $S_1^2$  and  $S_2^2$  are equal to  $s_1(s_1 + 1)\hbar^2 = \frac{3\hbar^2}{4}$ . For spin-1,  $s = 1$  and for the three basis states with  $m = \pm 1, 0$ , we have, since all operators are diagonal in this space:

(0.43)

$$\begin{aligned} (\mathbf{S}_1 \cdot \mathbf{S}_2) |s = 1, m = \pm 1, 0\rangle &= \frac{1}{2} (S^2 - S_1^2 - S_2^2) |s = 1, m = \pm 1, 0\rangle \\ (0.44) \qquad \qquad \qquad &= \frac{\hbar^2}{2} \left( s(s+1) - \frac{3}{2} \right) I |s = 1, m = \pm 1, 0\rangle \end{aligned}$$

$$(0.45) \qquad \qquad \qquad = \frac{\hbar^2}{4} |s = 1, m = \pm 1, 0\rangle$$

For the spin-0 state, there is only one basis state with  $m = 0$ , so

$$(0.46) \quad (\mathbf{S}_1 \cdot \mathbf{S}_2) |s = 0, m = 0\rangle = \frac{1}{2} (S^2 - S_1^2 - S_2^2) |s = 0, m = 0\rangle$$

$$(0.47) \qquad \qquad \qquad = \frac{\hbar^2}{2} \left( s(s+1) - \frac{3}{2} \right) |s = 0, m = 0\rangle$$

$$(0.48) \qquad \qquad \qquad = -\frac{3\hbar^2}{4} |s = 0, m = 0\rangle$$

Therefore, on any spin-1 state, we have

$$(0.49) \quad \mathbb{P}_1 |s = 1, m = \pm 1, 0\rangle = \left( \frac{3}{4}I + \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \right) |s = 1, m = \pm 1, 0\rangle$$

$$(0.50) \qquad \qquad \qquad = \left( \frac{3}{4} + \frac{1}{4} \right) I |s = 1, m = \pm 1, 0\rangle$$

$$(0.51) \qquad \qquad \qquad = |s = 1, m = \pm 1, 0\rangle$$

$$(0.52) \quad \mathbb{P}_2 |s = 1, m = \pm 1, 0\rangle = \left( \frac{1}{4}I - \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \right) |s = 1, m = \pm 1, 0\rangle$$

$$(0.53) \qquad \qquad \qquad = \left( \frac{1}{4} - \frac{1}{4} \right) I |s = 1, m = \pm 1, 0\rangle$$

$$(0.54) \qquad \qquad \qquad = 0$$

On the spin-0 state

$$(0.55) \quad \mathbb{P}_1 |s=0, m=0\rangle = \left( \frac{3}{4}I + \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \right) |s=0, m=0\rangle$$

$$(0.56) \quad = \left( \frac{3}{4} - \frac{3}{4} \right) I |s=0, m=0\rangle$$

$$(0.57) \quad = 0$$

$$(0.58) \quad \mathbb{P}_2 |s=0, m=0\rangle = \left( \frac{1}{4}I - \frac{1}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \right) |s=0, m=0\rangle$$

$$(0.59) \quad = \left( \frac{1}{4} + \frac{3}{4} \right) I |s=0, m=0\rangle$$

$$(0.60) \quad = |s=0, m=0\rangle$$

Since the four kets  $|s=1, m=\pm 1, 0\rangle$  and  $|s=0, m=0\rangle$  form a basis in the total- $j$  space, any state can be written as a linear combination of them, and thus the projection operator  $\mathbb{P}_1$  projects an arbitrary vector onto the  $|s=1, m=\pm 1, 0\rangle$  subspace and  $\mathbb{P}_2$  onto the  $|s=0, m=0\rangle$  subspace.

#### PINGBACKS

Pingback: Projection operators for general L + spin-1/2