PROJECTION OPERATORS FOR SPIN-1/2 + SPIN-1/2

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.5.

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We've seen projection operators in a formal mathematical sense, but in this post, we'll see a practical example of projection operators in spin space. We look at a system of two spin-$\frac{1}{2}$ particles, with spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$ for each of the two particles. Now consider the operators

$$\mathbb{P}_1 = \frac{3}{4}I+\frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2 \tag{1}$$

$$\mathbb{P}_2 = \frac{1}{4}I-\frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2 \tag{2}$$

A projection operator projects an arbitrary vector onto a subscape of the vector space in which that vector resides. The two projection operators here project onto orthogonal subspaces, which means if we project some vector V first with $\mathbb{P}_1$ and then with $\mathbb{P}_2$ (or vice versa), we'll end up with the zero vector. Also, if we project V twice (or more) with the same projection operator, all projections after the first have no further effect. That is

$$\mathbb{P}_i\mathbb{P}_j=\delta_{ij}\mathbb{P}_j \tag{3}$$

To show that this is true for the two projection operators above, we can make use of an identity derived earlier:

$$(\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma)=(\mathbf{A}\cdot\mathbf{B})I+i(\mathbf{A}\times\mathbf{B})\cdot\sigma \tag{4}$$

which is valid if $\mathbf{A}$ and $\mathbf{B}$ commute with σ.

Here $\mathbf{A}$ and $\mathbf{B}$ are vector operators that commute with the Pauli matrices σ.

First, we'll look at $\mathbb{P}_1\mathbb{P}_2$:

$$\mathbb{P}_1\mathbb{P}_2 = \left[\frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2\right]\left[\frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2\right] \tag{5}$$

$$= \left[\frac{3}{4}I + \frac{1}{4}\sigma_1\cdot\sigma_2\right]\left[\frac{1}{4}I - \frac{1}{4}\sigma_1\cdot\sigma_2\right] \tag{6}$$

$$= \frac{3}{16}I - \frac{2}{16}\sigma_1\cdot\sigma_2 - \frac{1}{16}\left(\sigma_1\cdot\sigma_2\right)^2 \tag{7}$$

We can write the last term as

$$\left(\sigma_1\cdot\sigma_2\right)^2 = \left(\sigma_1\cdot\sigma_2\right)\left(\sigma_1\cdot\sigma_2\right) \tag{8}$$

We see that this has the same form as 4 with $\mathbf{A} = \mathbf{B} = \sigma_1$ and $\sigma = \sigma_2$. Since σ_1 and σ_2 refer to different spins, they commute, so the identity is valid. We get

$$\left(\sigma_1\cdot\sigma_2\right)^2 = \sigma_1\cdot\sigma_1 I + i\left(\sigma_1\times\sigma_1\right)\cdot\sigma_2 \tag{9}$$

The first term is (using the fact that the square of each Pauli matrix is I):

$$\sigma_1\cdot\sigma_1 I = \left(\sigma_{x1}^2 + \sigma_{y1}^2 + \sigma_{z1}^2\right) I \tag{10}$$

$$= 3I^2 \tag{11}$$

$$= 3I \tag{12}$$

The cross product is just a shorthand way of writing the commutation relations. To see this, work out the x component, for example:

$$\left(\sigma_1\times\sigma_1\right)_x = \sigma_{y1}\sigma_{z1} - \sigma_{z1}\sigma_{y1} = 2i\sigma_{x1} \tag{13}$$

We can write this as

$$\left(\sigma_1\times\sigma_1\right) = i\sigma_1 \tag{14}$$

Plugging this into 9 we have

$$\left(\sigma_1\cdot\sigma_2\right)^2 = 3I - 2\sigma_1\cdot\sigma_2 \tag{15}$$

This gives, from 7

$$\mathbb{P}_1\mathbb{P}_2 = \frac{3}{16}I - \frac{2}{16}\sigma_1\cdot\sigma_2 - \frac{3}{16}I + \frac{2}{16}\sigma_1\cdot\sigma_2 = 0 \tag{16}$$

A similar calculation shows that

$$\mathbb{P}_2\mathbb{P}_1 = 0 \tag{17}$$

We can also calculate

$$\begin{aligned}
\mathbb{P}_1\mathbb{P}_1 &= \left[\frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2\right]\left[\frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2\right] && (18)\\
&= \left[\frac{3}{4}I + \frac{1}{4}\sigma_1\cdot\sigma_2\right]\left[\frac{3}{4}I + \frac{1}{4}\sigma_1\cdot\sigma_2\right] && (19)\\
&= \frac{9}{16}I + \frac{6}{16}\sigma_1\cdot\sigma_2 + \frac{1}{16}\left(\sigma_1\cdot\sigma_2\right)^2 && (20)\\
&= \frac{12}{16}I + \frac{4}{16}\sigma_1\cdot\sigma_2 && (21)\\
&= \frac{3}{4}I + \frac{1}{4}\sigma_1\cdot\sigma_2 && (22)\\
&= \mathbb{P}_1 && (23)
\end{aligned}$$

A similar calculation shows that

$$\mathbb{P}_2\mathbb{P}_2 = \mathbb{P}_2 \qquad (24)$$

To find the subspace to which each projection operator projects, we can use the explicit matrix forms in the product basis for the projections. We have

$$\begin{aligned}
\mathbb{P}_1 &= \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2 && (25)\\
&= \frac{3}{4}I + \frac{1}{\hbar^2}\left(\frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_{1-}S_{2+} + S_{1z}S_{2z}\right) && (26)\\
&= \frac{3}{4}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + && (27)\\
&\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} && (28)\\
&= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0 & 1\end{bmatrix} && (29)
\end{aligned}$$

Similarly

$$\mathbb{P}_2 = \frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{30}$$

$$= \frac{3}{4}I - \frac{1}{\hbar^2}\left(\frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_{1-}S_{2+} + S_{1z}S_{2z}\right) \tag{31}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \tag{32}$$

$$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

If we apply these projections to an arbitrary vector V, we have

$$\mathbb{P}_1 V = \mathbb{P}_1 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} a \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) \\ d \end{bmatrix} \tag{36}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}(b+c)\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{37}$$

Thus $\mathbb{P}_1$ projects V into the subspace spanned by the basis vectors of the 3-dimensional spin-1 subspace.

For $\mathbb{P}_2$ we have

$$\mathbb{P}_2 V = \mathbb{P}_2 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{2}(b-c) \\ \frac{1}{2}(-b+c) \\ 0 \end{bmatrix} \tag{39}$$

$$= \frac{1}{\sqrt{2}}(b-c) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{40}$$

Thus $\mathbb{P}_2$ projects onto the 1-dimensional spin-0 subspace.
In the total-j basis

$$S^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{41}$$

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) \tag{42}$$

In both the spin-1 and spin-0 states, the eigenvalues of S_1^2 and S_2^2 are equal to $s_1(s_1+1)\hbar^2 = \frac{3\hbar^2}{4}$. For spin-1, $s=1$ and for the three basis states with $m = \pm 1, 0$, we have, since all operators are diagonal in this space:

$$(\mathbf{S}_1 \cdot \mathbf{S}_2)|s=1, m=\pm 1, 0\rangle = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)|s=1, m=\pm 1, 0\rangle \tag{43}$$

$$= \frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right) I\,|s=1, m=\pm 1, 0\rangle \tag{44}$$

$$= \frac{\hbar^2}{4}|s=1, m=\pm 1, 0\rangle \tag{45}$$

For the spin-0 state, there is only one basis state with $m=0$, so

$$(\mathbf{S}_1 \cdot \mathbf{S}_2)\,|s=0,m=0\rangle = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)|s=0,m=0\rangle \tag{46}$$
$$= \frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right)|s=0,m=0\rangle \tag{47}$$
$$= -\frac{3\hbar^2}{4}\,|s=0,m=0\rangle \tag{48}$$

Therefore, on any spin-1 state, we have

$$\mathbb{P}_1\,|s=1,m=\pm1,0\rangle = \left(\frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2\right)|s=1,m=\pm1,0\rangle \tag{49}$$
$$= \left(\frac{3}{4} + \frac{1}{4}\right)I\,|s=1,m=\pm1,0\rangle \tag{50}$$
$$= |s=1,m=\pm1,0\rangle \tag{51}$$
$$\mathbb{P}_2\,|s=1,m=\pm1,0\rangle = \left(\frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2\right)|s=1,m=\pm1,0\rangle \tag{52}$$
$$= \left(\frac{1}{4} - \frac{1}{4}\right)I\,|s=1,m=\pm1,0\rangle \tag{53}$$
$$= 0 \tag{54}$$

On the spin-0 state

$$\mathbb{P}_1\,|s=0,m=0\rangle = \left(\frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2\right)|s=0,m=0\rangle \tag{55}$$
$$= \left(\frac{3}{4} - \frac{3}{4}\right)I\,|s=0,m=0\rangle \tag{56}$$
$$= 0 \tag{57}$$
$$\mathbb{P}_2\,|s=0,m=0\rangle = \left(\frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2\right)|s=0,m=0\rangle \tag{58}$$
$$= \left(\frac{1}{4} + \frac{3}{4}\right)I\,|s=0,m=0\rangle \tag{59}$$
$$= |s=0,m=0\rangle \tag{60}$$

Since the four kets $|s=1,m=\pm1,0\rangle$ and $|s=0,m=0\rangle$ form a basis in the total-j space, any state can be written as a linear combination of them, and thus the projection operator $\mathbb{P}_1$ projects an arbitrary vector onto the $|s=1,m=\pm1,0\rangle$ subspace and $\mathbb{P}_2$ onto the $|s=0,m=0\rangle$ subspace.

PINGBACKS

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