

## PROJECTION OPERATORS FOR GENERAL $\mathbf{L} + \text{SPIN-1/2}$

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.6.

We can generalize the calculation made earlier where we found the projection operators that project an arbitrary vector onto the spin-1 and spin-0 subspaces of the space where two spin- $\frac{1}{2}$  systems are added. Here, we'll consider adding a spin- $\frac{1}{2}$  system to a system with an arbitrary orbital angular momentum  $\mathbf{L}$ . In our earlier calculation, we found that the projection operators for adding two spin- $\frac{1}{2}$  systems are

$$\mathbb{P}_1 = \frac{3}{4}I + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \quad (1)$$

$$\mathbb{P}_2 = \frac{1}{4}I - \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \quad (2)$$

In the more general case, we'll assume that the projection operators have the forms

$$\mathbb{P}_+ = aI + \frac{b}{\hbar^2}\mathbf{L} \cdot \mathbf{S} \quad (3)$$

$$\mathbb{P}_- = cI + \frac{d}{\hbar^2}\mathbf{L} \cdot \mathbf{S} \quad (4)$$

where the constants  $a, b, c$  and  $d$  are to be determined. The operator  $\mathbb{P}_+$  should project a vector onto the  $j = l + \frac{1}{2}$  subspace and  $\mathbb{P}_-$  should project onto the  $j = l - \frac{1}{2}$  subspace. Consider  $\mathbb{P}_+$  first. We must therefore have

$$\mathbb{P}_+ \left| l + \frac{1}{2} \right\rangle = \left| l + \frac{1}{2} \right\rangle \quad (5)$$

First, we need a useful identity:

$$\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2 = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S} \quad (6)$$

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2) \quad (7)$$

Inserting 3 we have

$$\mathbb{P}_+ \left| l + \frac{1}{2} \right\rangle = \left( aI + \frac{b}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \left| l + \frac{1}{2} \right\rangle \quad (8)$$

$$= (a + b[j(j+1) - l(l+1) - s(s+1)]) \left| l + \frac{1}{2} \right\rangle \quad (9)$$

$$= \left( a + b \left[ \left( l + \frac{1}{2} \right) \left( l + \frac{3}{2} \right) - l(l+1) - \frac{3}{4} \right] \right) \left| l + \frac{1}{2} \right\rangle \quad (10)$$

$$= \left( a + \frac{bl}{2} \right) \left| l + \frac{1}{2} \right\rangle \quad (11)$$

Operating with  $\mathbb{P}_+$  on the state  $\left| l - \frac{1}{2} \right\rangle$  must give zero, since this state is orthogonal to  $\left| l + \frac{1}{2} \right\rangle$ , so

$$\mathbb{P}_+ \left| l - \frac{1}{2} \right\rangle = 0 \quad (12)$$

We therefore have

$$\mathbb{P}_+ \left| l - \frac{1}{2} \right\rangle = \left( aI + \frac{b}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \left| l - \frac{1}{2} \right\rangle \quad (13)$$

$$= (a + b[j(j+1) - l(l+1) - s(s+1)]) \left| l - \frac{1}{2} \right\rangle \quad (14)$$

$$= \left( a + b \left[ \left( l - \frac{1}{2} \right) \left( l + \frac{1}{2} \right) - l(l+1) - \frac{3}{4} \right] \right) \left| l - \frac{1}{2} \right\rangle \quad (15)$$

$$= \left( a - \frac{b(l+1)}{2} \right) \left| l - \frac{1}{2} \right\rangle \quad (16)$$

We thus have the two equations

$$a + \frac{bl}{2} = 1 \quad (17)$$

$$a - \frac{b(l+1)}{2} = 0 \quad (18)$$

Solving these, we find

$$a = \frac{l+1}{2l+1} \quad (19)$$

$$b = \frac{2}{2l+1} \quad (20)$$

The projection operator is therefore

$$\mathbb{P}_+ = \frac{1}{2l+1} \left[ (l+1)I + \frac{2}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right] \quad (21)$$

We can follow the same procedure to find  $\mathbb{P}_-$ . This yields the same results when we operate on the two states  $|l + \frac{1}{2}\rangle$  and  $|l - \frac{1}{2}\rangle$ , with  $a$  replaced by  $c$  and  $b$  by  $d$ , but now we require that

$$\mathbb{P}_- \left| l + \frac{1}{2} \right\rangle = 0 \quad (22)$$

$$\mathbb{P}_- \left| l - \frac{1}{2} \right\rangle = \left| l - \frac{1}{2} \right\rangle \quad (23)$$

This gives us the equations

$$c + \frac{dl}{2} = 0 \quad (24)$$

$$c - \frac{d(l+1)}{2} = 1 \quad (25)$$

with solutions

$$c = \frac{l}{2l+1} \quad (26)$$

$$d = -\frac{2}{2l+1} \quad (27)$$

Thus the projection operator is

$$\mathbb{P}_- = \frac{1}{2l+1} \left[ lI - \frac{2}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right] \quad (28)$$

As a check we see that 21 and 28 reduce to the correct forms 1 and 2 when  $l = \frac{1}{2}$ .