PROJECTION OPERATORS FOR GENERAL L + SPIN-1/2

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We can generalize the calculation made earlier where we found the projection operators that project an arbitrary vector onto the spin-1 and spin-0 subspaces of the space where two spin-$\frac{1}{2}$ systems are added. Here, we’ll consider adding a spin-$\frac{1}{2}$ system to a system with an arbitrary orbital angular momentum $L$. In our earlier calculation, we found that the projection operators for adding two spin-$\frac{1}{2}$ systems are

\[
P_1 = \frac{3}{4} I + \frac{1}{\hbar^2} S_1 \cdot S_2
\]

\[
P_2 = \frac{1}{4} I - \frac{1}{\hbar^2} S_1 \cdot S_2
\]

In the more general case, we’ll assume that the projection operators have the forms

\[
P_+ = a I + \frac{b}{\hbar^2} L \cdot S
\]

\[
P_- = c I + \frac{d}{\hbar^2} L \cdot S
\]

where the constants $a, b, c$ and $d$ are to be determined. The operator $P_+$ should project a vector onto the $j = l + \frac{1}{2}$ subspace and $P_-$ should project onto the $j = l - \frac{1}{2}$ subspace. Consider $P_+$ first. We must therefore have

\[
P_+ \left| l + \frac{1}{2} \right\rangle = \left| l + \frac{1}{2} \right\rangle
\]

First, we need a useful identity:

\[
J^2 = (L + S)^2 = L^2 + S^2 + 2 L \cdot S
\]

\[
L \cdot S = \frac{1}{2} (J^2 - L^2 - S^2)
\]

Inserting we have
\[ \mathbb{P}_+ \left| l + \frac{1}{2} \right\rangle = \left( aI + \frac{b}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \left| l + \frac{1}{2} \right\rangle \]  
(8)

\[ = (a + b[j (j + 1) - l (l + 1) - s (s + 1)]) \left| l + \frac{1}{2} \right\rangle \]  
(9)

\[ = \left( a + b \left[ \left( l + \frac{1}{2} \right) \left( l + \frac{3}{2} \right) - l (l + 1) - \frac{3}{4} \right] \right) \left| l + \frac{1}{2} \right\rangle \]  
(10)

\[ = \left( a + \frac{bl}{2} \right) \left| l + \frac{1}{2} \right\rangle \]  
(11)

Operating with \( \mathbb{P}_+ \) on the state \( \left| l - \frac{1}{2} \right\rangle \) must give zero, since this state is orthogonal to \( \left| l + \frac{1}{2} \right\rangle \), so

\[ \mathbb{P}_+ \left| l - \frac{1}{2} \right\rangle = 0 \]  
(12)

We therefore have

\[ \mathbb{P}_+ \left| l - \frac{1}{2} \right\rangle = \left( aI + \frac{b}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \left| l - \frac{1}{2} \right\rangle \]  
(13)

\[ = (a + b[j (j + 1) - l (l + 1) - s (s + 1)]) \left| l - \frac{1}{2} \right\rangle \]  
(14)

\[ = \left( a + b \left[ \left( l - \frac{1}{2} \right) \left( l + \frac{1}{2} \right) - l (l + 1) - \frac{3}{4} \right] \right) \left| l - \frac{1}{2} \right\rangle \]  
(15)

\[ = \left( a - \frac{b(l + 1)}{2} \right) \left| l - \frac{1}{2} \right\rangle \]  
(16)

We thus have the two equations

\[ a + \frac{bl}{2} = 1 \]  
(17)

\[ a - \frac{b(l + 1)}{2} = 0 \]  
(18)

Solving these, we find

\[ a = \frac{l + 1}{2l + 1} \]  
(19)

\[ b = \frac{2}{2l + 1} \]  
(20)

The projection operator is therefore
\[ \mathbb{P}_+ = \frac{1}{2l+1} \left( (l+1) I + \frac{2}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \]  \hspace{1cm} (21)

We can follow the same procedure to find \( \mathbb{P}_- \). This yields the same results when we operate on the two states \( |l + \frac{1}{2}\rangle \) and \( |l - \frac{1}{2}\rangle \), with \( a \) replaced by \( c \) and \( b \) by \( d \), but now we require that

\[ \mathbb{P}_- |l + \frac{1}{2}\rangle = 0 \] \hspace{1cm} (22)

\[ \mathbb{P}_- |l - \frac{1}{2}\rangle = |l - \frac{1}{2}\rangle \] \hspace{1cm} (23)

This gives us the equations

\[ c + \frac{dl}{2} = 0 \] \hspace{1cm} (24)

\[ c - \frac{d(l+1)}{2} = 1 \] \hspace{1cm} (25)

with solutions

\[ c = \frac{l}{2l+1} \] \hspace{1cm} (26)

\[ d = -\frac{2}{2l+1} \] \hspace{1cm} (27)

Thus the projection operator is

\[ \mathbb{P}_- = \frac{1}{2l+1} \left( I - \frac{2}{\hbar^2} \mathbf{L} \cdot \mathbf{S} \right) \] \hspace{1cm} (28)

As a check we see that (21) and (28) reduce to the correct forms (1) and (2) when \( l = \frac{1}{2} \).