PROJECTION OPERATORS FOR GENERAL L + SPIN-1/2

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.6.

We can generalize the calculation made earlier where we found the projection operators that project an arbitrary vector onto the spin-1 and spin-0 subspaces of the space where two spin-$\frac{1}{2}$ systems are added. Here, we'll consider adding a spin-$\frac{1}{2}$ system to a system with an arbitrary orbital angular momentum $\mathbf{L}$. In our earlier calculation, we found that the projection operators for adding two spin-$\frac{1}{2}$ systems are

$$\mathbb{P}_1 = \frac{3}{4}I+\frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2 \tag{1}$$

$$\mathbb{P}_2 = \frac{1}{4}I-\frac{1}{\hbar^2}\mathbf{S}_1\cdot\mathbf{S}_2 \tag{2}$$

In the more general case, we'll assume that the projection operators have the forms

$$\mathbb{P}_+ = aI+\frac{b}{\hbar^2}\mathbf{L}\cdot\mathbf{S} \tag{3}$$

$$\mathbb{P}_- = cI+\frac{d}{\hbar^2}\mathbf{L}\cdot\mathbf{S} \tag{4}$$

where the constants a,b,c and d are to be determined. The operator $\mathbb{P}_+$ should project a vector onto the $j=l+\frac{1}{2}$ subspace and $\mathbb{P}_-$ should project onto the $j=l-\frac{1}{2}$ subspace. Consider $\mathbb{P}_+$ first. We must therefore have

$$\mathbb{P}_+\left|l+\frac{1}{2}\right\rangle = \left|l+\frac{1}{2}\right\rangle \tag{5}$$

First, we need a useful identity:

$$\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2 = L^2+S^2+2\mathbf{L}\cdot\mathbf{S} \tag{6}$$

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}\left(J^2-L^2-S^2\right) \tag{7}$$

Inserting 3 we have

$$\mathbb{P}_+\left|l+\frac{1}{2}\right\rangle = \left(aI+\frac{b}{\hbar^2}\mathbf{L}\cdot\mathbf{S}\right)\left|l+\frac{1}{2}\right\rangle \tag{8}$$

$$= \left(a+b\left[j\left(j+1\right)-l\left(l+1\right)-s\left(s+1\right)\right]\right)\left|l+\frac{1}{2}\right\rangle \tag{9}$$

$$= \left(a+b\left[\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)-l\left(l+1\right)-\frac{3}{4}\right]\right)\left|l+\frac{1}{2}\right\rangle \tag{10}$$

$$= \left(a+\frac{bl}{2}\right)\left|l+\frac{1}{2}\right\rangle \tag{11}$$

Operating with $\mathbb{P}_+$ on the state $\left|l-\frac{1}{2}\right\rangle$ must give zero, since this state is orthogonal to $\left|l+\frac{1}{2}\right\rangle$, so

$$\mathbb{P}_+\left|l-\frac{1}{2}\right\rangle = 0 \tag{12}$$

We therefore have

$$\mathbb{P}_+\left|l-\frac{1}{2}\right\rangle = \left(aI+\frac{b}{\hbar^2}\mathbf{L}\cdot\mathbf{S}\right)\left|l-\frac{1}{2}\right\rangle \tag{13}$$

$$= \left(a+b\left[j\left(j+1\right)-l\left(l+1\right)-s\left(s+1\right)\right]\right)\left|l-\frac{1}{2}\right\rangle \tag{14}$$

$$= \left(a+b\left[\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right)-l\left(l+1\right)-\frac{3}{4}\right]\right)\left|l-\frac{1}{2}\right\rangle \tag{15}$$

$$= \left(a-\frac{b\left(l+1\right)}{2}\right)\left|l-\frac{1}{2}\right\rangle \tag{16}$$

We thus have the two equations

$$a+\frac{bl}{2} = 1 \tag{17}$$

$$a-\frac{b\left(l+1\right)}{2} = 0 \tag{18}$$

Solving these, we find

$$a = \frac{l+1}{2l+1} \tag{19}$$

$$b = \frac{2}{2l+1} \tag{20}$$

The projection operator is therefore

$$\mathbb{P}_+ = \frac{1}{2l+1}\left[(l+1)I + \frac{2}{\hbar^2}\mathbf{L}\cdot\mathbf{S}\right] \tag{21}$$

We can follow the same procedure to find $\mathbb{P}_-$. This yields the same results when we operate on the two states $\left|l+\frac{1}{2}\right\rangle$ and $\left|l-\frac{1}{2}\right\rangle$, with a replaced by c and b by d, but now we require that

$$\mathbb{P}_-\left|l+\frac{1}{2}\right\rangle = 0 \tag{22}$$

$$\mathbb{P}_-\left|l-\frac{1}{2}\right\rangle = \left|l-\frac{1}{2}\right\rangle \tag{23}$$

This gives us the equations

$$c + \frac{dl}{2} = 0 \tag{24}$$

$$c - \frac{d(l+1)}{2} = 1 \tag{25}$$

with solutions

$$c = \frac{l}{2l+1} \tag{26}$$

$$d = -\frac{2}{2l+1} \tag{27}$$

Thus the projection operator is

$$\mathbb{P}_- = \frac{1}{2l+1}\left[lI - \frac{2}{\hbar^2}\mathbf{L}\cdot\mathbf{S}\right] \tag{28}$$

As a check we see that 21 and 28 reduce to the correct forms 1 and 2when $l = \frac{1}{2}$.