

## SYMMETRY OF STATES FORMED FROM TWO EQUAL SPINS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.2; Exercise 15.2.7.

When we combine two states with angular momenta  $\mathbf{J}_1$  and  $\mathbf{J}_2$  we get a state with a total angular momentum  $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$  where the total angular momentum quantum number (that is, the angular momentum in units of  $\hbar$ ) can take on values from a maximum of  $j = j_1 + j_2$  down to either  $\frac{1}{2}$  or zero. The general expression uses the Clebsch-Gordan coefficients to write the compound state (in the total- $j$  space) as a linear combination of product states.

Although the calculation of Clebsch-Gordan coefficients can get quite complicated in the general case, we can explore a couple of interesting properties in the case where the two component spins (I'll refer to the angular momenta as spins for brevity, although the argument applies to the addition of angular momenta in general) are equal, so that  $j_1 = j_2$ .

[Note that in my edition of Shankar's book, there are a couple of typos in exercise 15.2.7, only one of which is corrected in the errata. The problem should ask us to show that states with  $j = 2j_1$  (not  $j = 2j_1 - 1$ ) are symmetric, and states with  $j = 2j_1 - 1$  are antisymmetric.]

We'll consider first the case where the  $z$ -component is maximum, so that  $m = 2j_1$ . In this case, we have

$$|2j_1, 2j_1\rangle_t = |j_1 j_1, j_1 j_1\rangle_p \quad (1)$$

The subscript  $t$  refers to a total- $j$  ket and  $p$  to a product ket. Thus the ket  $|2j_1, 2j_1\rangle_t$  is a total- $j$  ket with  $j = 2j_1$  and  $m = 2j_1$ , while  $|j_1 j_1, j_1 j_1\rangle_p$  is a product ket where both particles have total spin  $j_1$  and  $z$ -component  $j_1$ . Clearly if we swap the two particles in the product ket, it remains unchanged so this is a symmetric state. We would like to show that all states with  $j = 2j_1$  (that is, for all values of  $m$ ) are also symmetric.

To see this, we can follow Shankar's procedure of applying the lowering operator  $J_- = J_{I-} + J_{II-}$  to the original state. To clarify the notation, I've used roman numerals I and II to refer to the particle number, while the term  $j_1$  just refers to the (common) angular momentum number. It gets a bit confusing since in Shankar's formulas, the two particles were also assumed to have different angular momenta so he could use the subscripts 1 and 2

to refer both to the particle number and the angular momentum numbers. In our case, the two angular momenta are the same, but there are still two particles that we want to keep track of.

When the lowering operator is applied to 1 we get the state which can be obtained from Shankar's equation 15.2.7 with  $j_1 = j_2$ :

$$|2j_1, 2j_1 - 1\rangle = \frac{1}{\sqrt{2}} (|j_1(j_1 - 1), j_1 j_1\rangle + |j_1 j_1, j_1(j_1 - 1)\rangle) \quad (2)$$

By inspection, the state on the RHS is also symmetric when we swap the two particles. What happens if we carry on applying the lowering operator? To get an idea, consider the possible  $m_I$  and  $m_{II}$  values for each value of total  $z$ -component  $m$ .

$m$	$m_I, m_{II}$
$2j_1$	$j_1$
$2j_1 - 1$	$j_1, j_1 - 1$
$2j_1 - 2$	$j_1, j_1 - 1, j_1 - 2$
$2j_1 - a$	$j_1, j_1 - 1, \dots, j_1 - a$

In the last line, we've given the  $m_I$  and  $m_{II}$  values for an arbitrary value  $a$ , where  $a = 0, \dots, j_1$ . In each case, the ket  $|2j_1, 2j_1 - a\rangle$  is formed from a sum of product kets, where the two particles in each product ket must be chosen so that  $m_I + m_{II} = m$ . From the first line in the table we get 1 and from the second we get 2. If we apply the lowering operator  $J_- = J_{I-} + J_{II-}$  to 2, we can see the following pattern. Note that the two kets on the RHS are swapped versions of each other. If we apply  $J_{I-}$  to the first ket on the RHS and  $J_{II-}$  to the second ket, we get two kets that are again swapped versions of each other, except now one of the particles has a  $z$ -component of  $j_1$  and the other has  $j_1 - 2$ . These two kets are symmetric with respect to swapping.

Now if we apply  $J_{I-}$  to the *second* ket and  $J_{II-}$  to the first, we get two kets, in both of which the  $z$ -components of the two particles are both equal to  $j_1 - 1$ . Thus these two kets are again symmetric with respect to each other. (Actually, of course, we can combine them into a single ket of form  $|(j_1 - 1)(j_1 - 1), (j_1 - 1)(j_1 - 1)\rangle$  with some numerical coefficient, but for the purposes of our argument, it's better to keep them as two separate symmetric kets.)

We can see that the same pattern occurs as we continue the lowering process. In each case we apply  $J_{I-}$  to one member of a symmetric pair of kets and  $J_{II-}$  to the other member of the same pair. This always produces another pair of kets that are also symmetric. Thus the lowering process retains the symmetric property of the original ket 1 that was at the top of the column.

The lowering operators introduce numerical coefficients since they have the form

$$J_- |jmj_1j_2\rangle = \hbar\sqrt{(j+m)(j-m+1)} |j(m-1)j_1j_2\rangle \quad (3)$$

However, these coefficients will be the same for each pair of symmetric states. Let's do an explicit example to see how this works. Suppose we look at one symmetric pair in one line of the above table so we have the two kets

$$|j_1(m-b), j_1b\rangle + |j_1b, j_1(m-b)\rangle \quad (4)$$

We assume that these two kets have the same numerical coefficient at this stage. If we look at 2 we see this is true for the case  $b = 1$ , so we can take this as an anchor step in an inductive proof.

We now apply  $J_{I-}$  to the first ket and  $J_{II-}$  to the second ket in this pair. This gives

$$J_{I-} |j_1(m-b), j_1b\rangle + J_{II-} |j_1b, j_1(m-b)\rangle = \quad (5)$$

$$\hbar\sqrt{(j_1+m-b)(j_1-m+b+1)} [|j_1(m-b-1), j_1b\rangle + |j_1b, j_1(m-b-1)\rangle] \quad (6)$$

Thus the numerical coefficient is the same for both kets. The other symmetric pair is obtained by swapping the two lowering operators, so we have

$$J_{II-} |j_1(m-b), j_1b\rangle + J_{I-} |j_1b, j_1(m-b)\rangle = \quad (7)$$

$$\hbar\sqrt{(j_1+b)(j_1-b+1)} [|j_1(m-b), j_1(b-1)\rangle + J_{II-} |j_1(b-1), j_1(m-b)\rangle] \quad (8)$$

This results in another symmetric pair of kets.

Now consider states where  $j = 2j_1 - 1$ . The top member of this column can be obtained from Shankar's equation 15.2.8 with  $j_1 = j_2$ :

$$|2j_1 - 1, 2j_1 - 1\rangle = \frac{1}{\sqrt{2}} (|j_1j_1, j_1(j_1 - 1)\rangle - |j_1(j_1 - 1), j_1j_1\rangle) \quad (9)$$

Swapping the two particles on the RHS gives the negative of the original state, so this is an antisymmetric state. We can apply exactly the same argument as above to see that all states with lower values of  $m$  are also antisymmetric. In each case, we apply  $J_{I-}$  to one member of an antisymmetric pair and  $J_{II-}$  to the other, resulting in another antisymmetric pair with a value of  $m$  one lower than before. In this case, however, all states where

$m_I = m_{II}$  will cancel out, since such states must be their own negatives. This is just the Pauli exclusion principle for antisymmetric states.