

WIGNER-ECKART THEOREM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3.

We can write a rank-1 spherical tensor operator T_1^q in terms of a 3-d vector \mathbf{V} as follows:

$$T_1^{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}} \quad (1)$$

$$T_1^0 = V_z \quad (2)$$

Shankar provides the example of the position operator R_1^q in spherical coordinates, but (to me, at least) the example needs a bit of clarification. First, we see that R_1^q can be written in terms of the rectangular position coordinates as

$$R_1^{\pm 1} = \mp \frac{x \pm iy}{\sqrt{2}} \quad (3)$$

$$R_1^0 = z \quad (4)$$

We've seen earlier that the spherical harmonics can be written as

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{4\pi}} \frac{x \pm iy}{\sqrt{2}r} \quad (5)$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \quad (6)$$

Therefore, we have

$$R_1^q = \sqrt{\frac{4\pi}{3}} r Y_1^q \quad (7)$$

Now suppose we calculate the matrix elements of R_1^q in the basis of angular momentum eigenstates $|\alpha lm\rangle$. Here, we're assuming that the total angular momentum is orbital so $\mathbf{J} = \mathbf{L}$, and α represents quantities that depend on things other than angular momentum. If the potential in the Hamiltonian is spherically symmetric, then the wave function can be written as the product of a radial function $R_{\alpha l}(r)$ and a spherical harmonic $Y_l^m(\theta, \phi)$. The

radial function depends only on r , and its precise form depends on the potential function $V(r)$. The spherical harmonic depends only on the angular coordinates θ and ϕ , and is independent of the potential. Using these facts, we can write the matrix element as

$$\langle \alpha_2 l_2 m_2 | R_1^q | \alpha_1 l_1 m_1 \rangle = \int R_{\alpha_2 l_2}^* (Y_{l_2}^{m_2})^* \sqrt{\frac{4\pi}{3}} r Y_1^q R_{\alpha_1 l_1} Y_{l_1}^{m_1} r^2 dr d\Omega \quad (8)$$

In this equation, I've omitted the explicit functional dependences of the functions on the coordinates to save space, and $d\Omega$ is an increment of solid angle, so $d\Omega = \sin\theta d\theta d\phi$. This integral splits into the product of two separate integrals: one over r only and the other over angles only. That is

$$\langle \alpha_2 l_2 m_2 | R_1^q | \alpha_1 l_1 m_1 \rangle = \sqrt{\frac{4\pi}{3}} \int R_{\alpha_2 l_2}^* r R_{\alpha_1 l_1} r^2 dr \times \int (Y_{l_2}^{m_2})^* Y_1^q Y_{l_1}^{m_1} d\Omega \quad (9)$$

The first integral is known as the *reduced matrix element*, and is written as

$$\langle \alpha_2 l_2 || R_1 || \alpha_1 l_1 \rangle \equiv \sqrt{\frac{4\pi}{3}} \int R_{\alpha_2 l_2}^* r R_{\alpha_1 l_1} r^2 dr \quad (10)$$

Notice that this factor is independent of the tensor index q , which appears only in the angular integral. That is, the radial integral is the same for all 3 values of q .

The angular integral is written as

$$\langle l_2 m_2 | 1q, l_1 m_1 \rangle \quad (11)$$

Shankar claims that this is (up to a numerical factor independent of m_1, m_2 and q) a Clebsch-Gordan coefficient, although he doesn't derive this, so we'll accept it at this point.

This result is a special case of the more general *Wigner-Eckart theorem*, which states that, for any spherical tensor operator T_k^q , its matrix elements can be written as the product of two factors, one of which is the reduced matrix element. That is

$$\langle \alpha_2 j_2 m_2 | T_k^q | \alpha_1 j_1 m_1 \rangle = \langle \alpha_2 j_2 || T_k || \alpha_1 j_1 \rangle \langle j_2 m_2 | kq, j_1 m_1 \rangle \quad (12)$$

All the dependence of the matrix element on spatial orientation (that is, on θ and ϕ) is contained in the second factor, which can be written in terms of Clebsch-Gordan coefficients.

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