WIGNER-ECKART THEOREM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3.

We can write a rank-1 spherical tensor operator T_1^q in terms of a 3-d vector **V** as follows:

$$T_1^{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}} \tag{1}$$

$$T_1^0 = V_z \tag{2}$$

Shankar provides the example of the position operator R_1^q in spherical coordinates, but (to me, at least) the example needs a bit of clarification. First, we see that R_1^q can be written in terms of the rectangular position coordinates as

$$R_1^{\pm 1} = \mp \frac{x \pm iy}{\sqrt{2}} \tag{3}$$

$$R_1^0 = z \tag{4}$$

We've seen earlier that the spherical harmonics can be written as

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{4\pi}} \frac{x \pm iy}{\sqrt{2}r} \tag{5}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \tag{6}$$

Therefore, we have

$$R_1^q = \sqrt{\frac{4\pi}{3}} r Y_1^q \tag{7}$$

Now suppose we calculate the matrix elements of R_1^q in the basis of angular momentum eigenstates $|\alpha l m\rangle$. Here, we're assuming that the total angular momentum is orbital so $\mathbf{J} = \mathbf{L}$, and α represents quantities that depend on things other than angular momentum. If the potential in the Hamiltonian is spherically symmetric, then the wave function can be written as the product of a radial function $R_{\alpha l}(r)$ and a spherical harmonic $Y_l^m(\theta, \phi)$.

The radial function depends only on r, and its precise form depends on the potential function $V(r)$. The spherical harmonic depends only on the angular coordinates θ and ϕ, and is independent of the potential. Using these facts, we can write the matrix element as

$$\left\langle \alpha_2 l_2 m_2 \left| R_1^q \right| \alpha_1 l_1 m_1 \right\rangle = \int R_{\alpha_2 l_2}^* \left(Y_{l_2}^{m_2} \right)^* \sqrt{\frac{4\pi}{3}} r Y_1^q R_{\alpha_1 l_1} Y_{l_1}^{m_1} r^2 dr\, d\Omega \tag{8}$$

In this equation, I've omitted the explicit functional dependences of the functions on the coordinates to save space, and $d\Omega$ is an increment of solid angle, so $d\Omega = \sin\theta d\theta\, d\phi$. This integral splits into the product of two separate integrals: one over r only and the other over angles only. That is

$$\left\langle \alpha_2 l_2 m_2 \left| R_1^q \right| \alpha_1 l_1 m_1 \right\rangle = \sqrt{\frac{4\pi}{3}} \int R_{\alpha_2 l_2}^* r R_{\alpha_1 l_1} r^2 dr \times \int \left(Y_{l_2}^{m_2} \right)^* Y_1^q Y_{l_1}^{m_1} d\Omega \tag{9}$$

The first integral is known as the *reduced matrix element*, and is written as

$$\langle \alpha_2 l_2 \| R_1 \| \alpha_1 l_1 \rangle \equiv \sqrt{\frac{4\pi}{3}} \int R_{\alpha_2 l_2}^* r R_{\alpha_1 l_1} r^2 dr \tag{10}$$

Notice that this factor is independent of the tensor index q, which appears only in the angular integral. That is, the radial integral is the same for all 3 values of q.

The angular integral is written as

$$\langle l_2 m_2 | 1q, l_1 m_1 \rangle \tag{11}$$

Shankar claims that this is (up to a numerical factor independent of m_1, m_2 and q) a Clebsch-Gordan coefficient, although he doesn't derive this, so we'll accept it at this point.

This result is a special case of the more general *Wigner-Eckart theorem*, which states that, for any spherical tensor operator T_k^q, its matrix elements can be written as the product of two factors, one of which is the reduced matrix element. That is

$$\left\langle \alpha_2 j_2 m_2 \left| T_k^q \right| \alpha_1 j_1 m_1 \right\rangle = \langle \alpha_2 j_2 \| T_k \| \alpha_1 j_1 \rangle \langle j_2 m_2 | kq, j_1 m_1 \rangle \tag{12}$$

All the dependence of the matrix element on spatial orientation (that is, on θ and ϕ) is contained in the second factor, which can be written in terms of Clebsch-Gordan coefficients.

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