

SPHERICAL TENSOR OPERATORS; COMMUTATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.1.

A *spherical tensor operator* is defined to be an object T_k^q with integer indices q and k . The rank of the tensor is k , and the other index q ranges in integer steps from $-k$ to $+k$, giving T_k^q $2k + 1$ components. Its definition includes a requirement that it transform under a rotation according to

$$(1) \quad U [R] T_k^q U^\dagger [R] = \sum_{q'} D_{q'q}^{(k)} T_k^{q'}$$

where $D^{(k)}$ is the k -th block in the block diagonal matrix formed from the angular momentum operators J . For a rotation through an angle θ about an axis $\hat{\theta}$, we have

$$(2) \quad D^{(k)} [R(\theta)] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\theta}{\hbar} \right)^n (\hat{\theta} \cdot \mathbf{J}^{(k)})^n$$

where $\mathbf{J}^{(k)}$ is the angular momentum vector obtained from the k -th block in each of J_x , J_y and J_z (see Shankar section 12.5 for details).

The series can be written in closed form for some small values of k , but we won't need these forms here.

For a set of angular momentum kets $|kq\rangle$ (Shankar changes the notation here, in that $|kq\rangle$ refers to a state with total angular momentum number k and z component q , rather than the more familiar $|jm\rangle$), the matrix elements of $D^{(k)}$ are

$$(3) \quad D_{q'q}^{(k)} = \langle kq' | U [R] | kq \rangle$$

Note that

$$(4) \quad \langle k'q' | U [R] | kq \rangle = D_{q'q}^{(k)} \delta_{k'k}$$

This follows because a rotation cannot change the total angular momentum of a state, so $U [R] |kq\rangle$ will always result in a state whose total angular

momentum number is also k . From this fact, we can write the rotation of an angular momentum ket as

$$(5) \quad U[R] |kq\rangle = \sum_{k'} \sum_{q'} |k'q'\rangle \langle k'q' | U[R] |kq\rangle$$

$$(6) \quad = \sum_{k'} \sum_{q'} |k'q'\rangle D_{q'q}^{(k)} \delta_{k'k}$$

$$(7) \quad = \sum_{q'} D_{q'q}^{(k)} |kq'\rangle$$

Comparing this result with 1, we see that a passive transformation of the tensor operator T_k^q works in the same way as a rotation of an angular momentum eigenstate $|kq\rangle$.

We can use 1 to work out the commutators of T_k^q with the components of the angular momentum operator \mathbf{J} . We use the fact that angular momentum is the generator of rotations and consider an infinitesimal rotation $\delta\theta$ about, say, the x axis. In this case, working to first order in $\delta\theta$:

$$(8) \quad U[R] = I - \frac{i\delta\theta J_x}{\hbar}$$

$$(9) \quad U^\dagger[R] = I + \frac{i\delta\theta J_x}{\hbar}$$

$$(10) \quad U[R] T_k^q U^\dagger[R] = \left(I - \frac{i\delta\theta J_x}{\hbar} \right) T_k^q \left(I + \frac{i\delta\theta J_x}{\hbar} \right)$$

$$(11) \quad = T_k^q - \frac{i\delta\theta}{\hbar} [J_x, T_k^q]$$

On the RHS of 1 we can use 3 to first order in $\delta\theta$:

$$(12) \quad D_{q'q}^{(k)} T_k^{q'} = \left\langle kq' \left| I - \frac{i\delta\theta J_x}{\hbar} \right| kq \right\rangle T_k^{q'}$$

$$(13) \quad = \langle kq' | kq \rangle T_k^{q'} - \frac{i\delta\theta}{\hbar} \langle kq' | J_x | kq \rangle T_k^{q'}$$

$$(14) \quad = T_k^{q'} - \frac{i\delta\theta}{\hbar} \langle kq' | J_x | kq \rangle T_k^{q'}$$

Combining the last two results, we have

$$(15) \quad [J_x, T_k^q] = \sum_{q'} \langle kq' | J_x | kq \rangle T_k^{q'}$$

We could do the same analysis for the y and z components, and we'd get the same result, so we have

$$(16) \quad [J_y, T_k^q] = \sum_{q'} \langle kq' | J_y | kq \rangle T_k^{q'}$$

$$(17) \quad [J_z, T_k^q] = \sum_{q'} \langle kq' | J_z | kq \rangle T_k^{q'}$$

We can simplify the last equation, since the ket $|kq\rangle$ is an eigenket of J_z with eigenvalue $q\hbar$. We therefore have

$$(18) \quad \sum_{q'} \langle kq' | J_z | kq \rangle T_k^{q'} = \sum_{q'} \langle kq' | kq \rangle \hbar q T_k^{q'}$$

$$(19) \quad = \hbar q T_k^q$$

To deal with the other two components, we can combine the results in 15 and 16 and use the raising and lowering operators.

$$(20) \quad J_{\pm} = J_x \pm iJ_y$$

$$(21) \quad J_{\pm} |kq\rangle = \hbar \sqrt{(k \mp q)(k \pm q + 1)} |k, q \pm 1\rangle$$

We have

$$(22) \quad [J_{\pm}, T_k^q] = \sum_{q'} \langle kq' | J_{\pm} | kq \rangle T_k^{q'}$$

$$(23) \quad = \hbar \sqrt{(k \mp q)(k \pm q + 1)} \sum_{q'} \langle kq' | k, q \pm 1 \rangle T_k^{q'}$$

$$(24) \quad = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_k^{q \pm 1}$$

where we've again used the orthogonality of the eigenkets to get the last line.

Example. Suppose we construct a spherical tensor out of the components of a vector operator \mathbf{V} so that we have a rank 1 tensor given by

$$(25) \quad T_1^{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}$$

$$(26) \quad T_1^0 = V_z$$

Vector operators obey the commutation rules

$$(27) \quad [V_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} V_k$$

Applying this gives us, for example

$$(28) \quad [T_1^1, J_x] = -\frac{1}{\sqrt{2}} ([V_x, J_x] + i[V_y, J_x])$$

$$(29) \quad = -\frac{1}{\sqrt{2}} (0 + \hbar V_z)$$

$$(30) \quad = -\hbar \frac{V_z}{\sqrt{2}}$$

$$(31) \quad [T_1^1, J_y] = -\frac{1}{\sqrt{2}} ([V_x, J_y] + i[V_y, J_y])$$

$$(32) \quad = -\frac{1}{\sqrt{2}} (i\hbar V_z + 0)$$

$$(33) \quad = -i\hbar \frac{V_z}{\sqrt{2}}$$

Combining these results, we have

$$(34) \quad [T_1^1, J_+] = [T_1^1, J_x] + i[T_1^1, J_y]$$

$$(35) \quad = -\hbar \frac{V_z}{\sqrt{2}} + \hbar \frac{V_z}{\sqrt{2}}$$

$$(36) \quad = 0$$

This agrees with 24 with $k = q = 1$.

We also have

$$(37) \quad [T_1^1, J_-] = [T_1^1, J_x] - i[T_1^1, J_y]$$

$$(38) \quad = -\hbar \frac{V_z}{\sqrt{2}} - \hbar \frac{V_z}{\sqrt{2}}$$

$$(39) \quad = -\sqrt{2}\hbar V_z$$

$$(40) \quad = -\sqrt{2}\hbar T_1^0$$

This also agrees with 24 with $k = q = 1$ (since $[T_1^1, J_-] = -[J_-, T_1^1]$).

We can do similar calculations to find that

$$(41) \quad [T_1^{-1}, J_+] = -\sqrt{2}\hbar T_1^0$$

$$(42) \quad [T_1^{-1}, J_-] = 0$$

Finally, we have

$$(43) \quad [T_1^1, J_z] = -\frac{1}{\sqrt{2}} ([V_x, J_z] + i[V_y, J_z])$$

$$(44) \quad = -\frac{1}{\sqrt{2}} (-i\hbar V_y - \hbar V_x)$$

$$(45) \quad = \frac{\hbar}{\sqrt{2}} (V_x + iV_y)$$

$$(46) \quad = -\hbar T_1^1$$

$$(47) \quad [J_z, T_1^1] = \hbar T_1^1$$

which is again consistent with 19 with $q = 1$. Similar calculations can be done to verify the other commutation relations.

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