# SPHERICAL TENSOR OPERATORS; COMMUTATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.1.

A *spherical tensor operator* is defined to be an object $T_k^q$ with integer indices $q$ and $k$. The rank of the tensor is $k$, and the other index $q$ ranges in integer steps from $-k$ to $+k$, giving $T_k^q$ $2k+1$ components. Its definition includes a requirement that it transform under a rotation according to

$$U[R]T_k^q U^\dagger[R]=\sum_{q'}D_{q'q}^{(k)}T_k^{q'} \tag{1}$$

where $D^{(k)}$ is the $k$-th block in the block diagonal matrix formed from the angular momentum operators $J$. For a rotation through an angle $\theta$ about an axis $\hat{\theta}$, we have

$$D^{(k)}[R(\theta)]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\theta}{\hbar}\right)^n\left(\hat{\theta}\cdot\mathbf{J}^{(k)}\right)^n \tag{2}$$

where $\mathbf{J}^{(k)}$ is the angular momentum vector obtained from the $k$-th block in each of $J_x$, $J_y$ and $J_z$ (see Shankar section 12.5 for details).

The series can be written in closed form for some small values of $k$, but we won't need these forms here.

For a set of angular momentum kets $|kq\rangle$ (Shankar changes the notation here, in that $|kq\rangle$ refers to a state with total angular momentum number $k$ and $z$ component $q$, rather than the more familiar $|jm\rangle$), the matrix elements of $D^{(k)}$ are

$$D_{q'q}^{(k)}=\left\langle kq'\left|U[R]\right|kq\right\rangle \tag{3}$$

Note that

$$\left\langle k'q'\left|U[R]\right|kq\right\rangle=D_{q'q}^{(k)}\delta_{k'k} \tag{4}$$

This follows because a rotation cannot change the total angular momentum of a state, so $U[R]|kq\rangle$ will always result in a state whose total angular momentum number is also $k$. From this fact, we can write the rotation of an angular momentum ket as

$$U[R]|kq\rangle = \sum_{k'}\sum_{q'}\left|k'q'\right\rangle\left\langle k'q'\left|U[R]\right|kq\right\rangle \tag{5}$$

$$= \sum_{k'}\sum_{q'}\left|k'q'\right\rangle D_{q'q}^{(k)}\delta_{k'k} \tag{6}$$

$$= \sum_{q'} D_{q'q}^{(k)}\left|kq'\right\rangle \tag{7}$$

Comparing this result with 1, we see that a passive transformation of the tensor operator $T_k^q$ works in the same way as a rotation of an angular momentum eigenstate $|kq\rangle$.

We can use 1 to work out the commutators of $T_k^q$ with the components of the angular momentum operator **J**. We use the fact that angular momentum is the generator of rotations and consider an infinitesimal rotation $\delta\theta$ about, say, the $x$ axis. In this case, working to first order in $\delta\theta$:

$$U[R] = I - \frac{i\delta\theta J_x}{\hbar} \tag{8}$$

$$U^\dagger[R] = I + \frac{i\delta\theta J_x}{\hbar} \tag{9}$$

$$U[R]T_k^q U^\dagger[R] = \left(I - \frac{i\delta\theta J_x}{\hbar}\right)T_k^q\left(I + \frac{i\delta\theta J_x}{\hbar}\right) \tag{10}$$

$$= T_k^q - \frac{i\delta\theta}{\hbar}\left[J_x, T_k^q\right] \tag{11}$$

On the RHS of 1 we can use 3 to first order in $\delta\theta$:

$$D_{q'q}^{(k)}T_k^{q'} = \left\langle kq'\left|I - \frac{i\delta\theta J_x}{\hbar}\right|kq\right\rangle T_k^{q'} \tag{12}$$

$$= \left\langle kq'\left|kq\right|\right\rangle T_k^{q'} - \frac{i\delta\theta}{\hbar}\left\langle kq'\left|J_x\right|kq\right\rangle T_k^{q'} \tag{13}$$

$$= T_k^q - \frac{i\delta\theta}{\hbar}\left\langle kq'\left|J_x\right|kq\right\rangle T_k^{q'} \tag{14}$$

Combining the last two results, we have

$$\left[J_x, T_k^q\right] = \sum_{q'}\left\langle kq'\left|J_x\right|kq\right\rangle T_k^{q'} \tag{15}$$

We could do the same analysis for the $y$ and $z$ components, and we'd get the same result, so we have

$$\left[J_y, T_k^q\right] = \sum_{q'} \left\langle kq' \left| J_y \right| kq \right\rangle T_k^{q'} \tag{16}$$

$$\left[J_z, T_k^q\right] = \sum_{q'} \left\langle kq' \left| J_z \right| kq \right\rangle T_k^{q'} \tag{17}$$

We can simplify the last equation, since the ket $|kq\rangle$ is an eigenket of $J_z$ with eigenvalue $q\hbar$. We therefore have

$$\sum_{q'} \left\langle kq' \left| J_z \right| kq \right\rangle T_k^{q'} = \sum_{q'} \left\langle kq' \left| kq \right.\right\rangle \hbar q T_k^{q'} \tag{18}$$

$$= \hbar q T_k^q \tag{19}$$

To deal with the other two components, we can combine the results in 15 and 16 and use the raising and lowering operators.

$$J_\pm = J_x \pm i J_y \tag{20}$$

$$J_\pm |kq\rangle = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\,|k, q \pm 1\rangle \tag{21}$$

We have

$$\left[J_\pm, T_k^q\right] = \sum_{q'} \left\langle kq' \left| J_\pm \right| kq \right\rangle T_k^{q'} \tag{22}$$

$$= \hbar\sqrt{(k \mp q)(k \pm q + 1)} \sum_{q'} \left\langle kq' \left| k, q \pm 1 \right.\right\rangle T_k^{q'} \tag{23}$$

$$= \hbar\sqrt{(k \mp q)(k \pm q + 1)}\, T_k^{q \pm 1} \tag{24}$$

where we've again used the orthogonality of the eigenkets to get the last line.

**Example.** Suppose we construct a spherical tensor out of the components of a vector operator **V** so that we have a rank 1 tensor given by

$$T_1^{\pm 1} = \mp \frac{V_x \pm i V_y}{\sqrt{2}} \tag{25}$$

$$T_1^0 = V_z \tag{26}$$

Vector operators obey the commutation rules

$$[V_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} V_k \tag{27}$$

Applying this gives us, for example

$$\left[T_1^1, J_x\right] = -\frac{1}{\sqrt{2}}\left([V_x, J_x] + i[V_y, J_x]\right) \quad (28)$$
$$= -\frac{1}{\sqrt{2}}\left(0 + \hbar V_z\right) \quad (29)$$
$$= -\hbar\frac{V_z}{\sqrt{2}} \quad (30)$$
$$\left[T_1^1, J_y\right] = -\frac{1}{\sqrt{2}}\left([V_x, J_y] + i[V_y, J_y]\right) \quad (31)$$
$$= -\frac{1}{\sqrt{2}}\left(i\hbar V_z + 0\right) \quad (32)$$
$$= -i\hbar\frac{V_z}{\sqrt{2}} \quad (33)$$

Combining these results, we have

$$\left[T_1^1, J_+\right] = \left[T_1^1, J_x\right] + i\left[T_1^1, J_y\right] \quad (34)$$
$$= -\hbar\frac{V_z}{\sqrt{2}} + \hbar\frac{V_z}{\sqrt{2}} \quad (35)$$
$$= 0 \quad (36)$$

This agrees with 24 with $k = q = 1$.
We also have

$$\left[T_1^1, J_-\right] = \left[T_1^1, J_x\right] - i\left[T_1^1, J_y\right] \quad (37)$$
$$= -\hbar\frac{V_z}{\sqrt{2}} - \hbar\frac{V_z}{\sqrt{2}} \quad (38)$$
$$= -\sqrt{2}\hbar V_z \quad (39)$$
$$= -\sqrt{2}\hbar T_1^0 \quad (40)$$

This also agrees with 24 with $k = q = 1$ (since $\left[T_1^1, J_-\right] = -\left[J_-, T_1^1\right]$).
We can do similar calculations to find that

$$\left[T_1^{-1}, J_+\right] = -\sqrt{2}\hbar T_1^0 \quad (41)$$
$$\left[T_1^{-1}, J_-\right] = 0 \quad (42)$$

Finally, we have

$$\left[T_1^1, J_z\right] = -\frac{1}{\sqrt{2}}\left(\left[V_x, J_z\right] + i\left[V_y, J_z\right]\right) \tag{43}$$

$$= -\frac{1}{\sqrt{2}}\left(-i\hbar V_y - \hbar V_x\right) \tag{44}$$

$$= \frac{\hbar}{\sqrt{2}}\left(V_x + iV_y\right) \tag{45}$$

$$= -\hbar T_1^1 \tag{46}$$

$$\left[J_z, T_1^1\right] = \hbar T_1^1 \tag{47}$$

which is again consistent with 19 with $q = 1$. Similar calculations can be done to verify the other commutation relations.

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