A spherical tensor operator is defined to be an object $T^q_k$ with integer indices $q$ and $k$. The rank of the tensor is $k$, and the other index $q$ ranges in integer steps from $-k$ to $+k$, giving $T_k^q 2k + 1$ components. Its definition includes a requirement that it transform under a rotation according to

$$ U[R] T^q_k U^\dagger[R] = \sum_{q'} D^{(k)}_{q'q} T^{q'}_k $$

where $D^{(k)}$ is the $k$-th block in the block diagonal matrix formed from the angular momentum operators $J$. For a rotation through an angle $\theta$ about an axis \( \hat{\theta} \), we have

$$ D^{(k)} [R(\theta)] = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i\theta}{\hbar} \right)^n \left( \hat{\theta} \cdot J^{(k)} \right)^n $$

where $J^{(k)}$ is the angular momentum vector obtained from the $k$-th block in each of $J_x$, $J_y$ and $J_z$ (see Shankar section 12.5 for details).

The series can be written in closed form for some small values of $k$, but we won’t need these forms here.

For a set of angular momentum kets $|kq\rangle$ (Shankar changes the notation here, in that $|kq\rangle$ refers to a state with total angular momentum number $k$ and $z$ component $q$, rather than the more familiar $|jm\rangle$), the matrix elements of $D^{(k)}$ are

$$ D^{(k)}_{q'q} = \langle kq' | U[R] | kq \rangle $$

Note that

$$ \langle k'q' | U[R] | kq \rangle = D^{(k)}_{q'q} \delta_{k'k} $$

This follows because a rotation cannot change the total angular momentum of a state, so $U[R] |kq\rangle$ will always result in a state whose total angular momentum number is also $k$. From this fact, we can write the rotation of an angular momentum ket as
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\[ U [R] |kq\rangle = \sum_{k'} \sum_{q'} |k'q'\rangle \langle k'q' | U [R] |kq\rangle \]
\[ = \sum_{k'} \sum_{q'} |k'q'\rangle D^{(k)}_{q'q} \delta_{k'k} \]
\[ = \sum_{q'} D^{(k)}_{q'q} |kq'\rangle \]

(5)

(6)

(7)

Comparing this result with (1) we see that a passive transformation of the tensor operator \( T^q_k \) works in the same way as a rotation of an angular momentum eigenstate \(|kq\rangle\).

We can use (1) to work out the commutators of \( T^q_k \) with the components of the angular momentum operator \( J \). We use the fact that angular momentum is the generator of rotations and consider an infinitesimal rotation \( \delta\theta \) about, say, the \( x \) axis. In this case, working to first order in \( \delta\theta \):

\[ U [R] = I - \frac{i\delta\theta J_x}{\hbar} \]
\[ U^\dagger [R] = I + \frac{i\delta\theta J_x}{\hbar} \]
\[ U [R] T^q_k U^\dagger [R] = \left( I - \frac{i\delta\theta J_x}{\hbar} \right) T^q_k \left( I + \frac{i\delta\theta J_x}{\hbar} \right) \]
\[ = T^q_k - \frac{i\delta\theta}{\hbar} [J_x, T^q_k] \]

(8)

(9)

(10)

(11)

On the RHS of (11) we can use (3) to first order in \( \delta\theta \):

\[ D^{(k)}_{q'q} T^q_k = \langle kq' | I - \frac{i\delta\theta J_x}{\hbar} | kq \rangle T^q_k \]
\[ = \langle kq' | kq \rangle T^q_k - \frac{i\delta\theta}{\hbar} \langle kq' | J_x | kq \rangle T^q_k \]
\[ = T^q_k - \frac{i\delta\theta}{\hbar} \langle kq' | J_x | kq \rangle T^q_k \]

(12)

(13)

(14)

Combining the last two results, we have

\[ [J_x, T^q_k] = \sum_{q'} \langle kq' | J_x | kq \rangle T^q_k \]

(15)

We could do the same analysis for the \( y \) and \( z \) components, and we’d get the same result, so we have
We can simplify the last equation, since the ket $|kq\rangle$ is an eigenket of $J_z$ with eigenvalue $q\hbar$. We therefore have

$$\sum_{q'} \langle kq' | J_z | kq \rangle T_{q'}^{q} = \sum_{q'} \langle kq' | kq \rangle \hbar q T_{q'}^{q} = \hbar q T_{k}^{q}$$

To deal with the other two components, we can combine the results in 15 and 16 and use the raising and lowering operators

$$J_{\pm} = J_x \pm iJ_y$$

$$J_{\pm} |kq\rangle = \hbar \sqrt{(k \mp q)(k \pm q + 1)} |k, q \pm 1\rangle$$

We have

$$[J_{\pm}, T_{k}^{q}] = \sum_{q'} \langle kq' | J_{\pm} | kq \rangle T_{q'}^{q}$$

$$= \hbar \sqrt{(k \mp q)(k \pm q + 1)} \sum_{q'} \langle kq' | k, q \pm 1 \rangle T_{q'}^{q}$$

$$= \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_{k}^{q \pm 1}$$

where we’ve again used the orthogonality of the eigenkets to get the last line.

**Example.** Suppose we construct a spherical tensor out of the components of a vector operator $V$ so that we have a rank 1 tensor given by

$$T_{i}^{\pm 1} = \frac{\mp V_x \pm iV_y}{\sqrt{2}}$$

$$T_{i}^{0} = V_z$$

Vector operators obey the commutation rules

$$[V_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} V_k$$

Applying this gives us, for example
$$[T^1_1, J_x] = -\frac{1}{\sqrt{2}} ([V_x, J_x] + i [V_y, J_x])$$
$$= -\frac{1}{\sqrt{2}} (0 + \hbar V_z)$$
$$= -\hbar \frac{V_z}{\sqrt{2}}$$
$$[T^1_1, J_y] = -\frac{1}{\sqrt{2}} ([V_x, J_y] + i [V_y, J_y])$$
$$= -\frac{1}{\sqrt{2}} (i \hbar V_z + 0)$$
$$= -i \hbar \frac{V_z}{\sqrt{2}}$$

Combining these results, we have

$$[T^1_1, J_+] = [T^1_1, J_x] + i [T^1_1, J_y]$$
$$= -\hbar \frac{V_z}{\sqrt{2}} + \hbar \frac{V_z}{\sqrt{2}}$$
$$= 0$$

This agrees with 24 with $k = q = 1$.

We also have

$$[T^1_1, J_-] = [T^1_1, J_x] - i [T^1_1, J_y]$$
$$= -\hbar \frac{V_z}{\sqrt{2}} - \hbar \frac{V_z}{\sqrt{2}}$$
$$= -\sqrt{2} \hbar V_z$$
$$= -\sqrt{2} \hbar T^0_1$$

This also agrees with 24 with $k = q = 1$ (since $[T^1_1, J_-] = -[J_-, T^1_1]$).

We can do similar calculations to find that

$$[T^{-1}_1, J_+] = -\sqrt{2} \hbar T^0_1$$
$$[T^{-1}_1, J_-] = 0$$

Finally, we have
\[ [T^1_1, J_z] = -\frac{1}{\sqrt{2}} ([V_x, J_z] + i [V_y, J_z]) \]  \hspace{1cm} (43)

\[ = -\frac{1}{\sqrt{2}} (-i\hbar V_y - \hbar V_x) \]  \hspace{1cm} (44)

\[ = \frac{\hbar}{\sqrt{2}} (V_x + iV_y) \]  \hspace{1cm} (45)

\[ = -\hbar T^1_1 \]  \hspace{1cm} (46)

\[ [J_z, T^1_1] = \hbar T^1_1 \]  \hspace{1cm} (47)

which is again consistent with 19 with \( q = 1 \). Similar calculations can be done to verify the other commutation relations.