SPHERICAL TENSOR OPERATORS; A SCALAR OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.2.

A spherical tensor operator is defined by the way it transforms under rotation:

$$U[R]\,T_k^q U^\dagger[R] = \sum_{q'} D_{q'q}^{(k)} T_k^{q'} \tag{1}$$

where $D^{(k)}$ is the k-th block in the block diagonal matrix formed from the angular momentum operators J. We can form an operator Ω from two spherical tensor operators:

$$\Omega_k \equiv \sum_q (-1)^q S_k^q T_k^{-q} \tag{2}$$

For $k=1$ we can write a spherical tensor operator in terms of a 3-d vector operator. We'll use lower-case letters to represent the vector operator, so we have

$$S_1^{\pm 1} = \mp \frac{s_x \pm i s_y}{\sqrt{2}} \tag{3}$$

$$S_1^0 = s_z \tag{4}$$

$$T_1^{\pm 1} = \mp \frac{t_x \pm i t_y}{\sqrt{2}} \tag{5}$$

$$T_1^0 = t_z \tag{6}$$

Plugging these into 2 with $k=1$ we have

$$\begin{aligned}
\Omega_1 &= \frac{1}{2}(s_x + i s_y)(t_x - i t_y) + s_z t_z + \frac{1}{2}(s_x - i s_y)(t_x + i t_y) && (7)\\
&= s_x t_x + s_y t_y + s_z t_z && (8)\\
&= \mathbf{s}\cdot\mathbf{t} && (9)
\end{aligned}$$

Thus Ω_1 is the scalar product of the two vectors, and is therefore a scalar operator.

To prove this for any k, we can let Ω_k operate on an angular momentum eigenket $|jm\rangle$ and then rotate this state, using 1. To simplify things, I'll write the unitary rotation operator $U[R]$ without the explicit R dependence, so it's just U.

$$U\Omega_k|jm\rangle = U\sum_q (-1)^q S_k^q T_k^{-q}|jm\rangle \tag{10}$$

$$= \sum_q (-1)^q US_k^q U^\dagger U T_k^{-q} U^\dagger U|jm\rangle \tag{11}$$

$$= \sum_{q,a,b,c} (-1)^q D_{aq}^{(k)} S_k^a D_{-b,-q}^{(k)} T_k^{-b} D_{cm}^{(j)}|jc\rangle \tag{12}$$

Each of the lower indices in $D_{m,m'}^{(k)}$ can take values $-k,\ldots,+k$, so a sum over m is the same as a sum over $-m$. That is

$$UT_k^{-q}U^\dagger = \sum_b D_{b,-q}^{(k)} T_k^b = \sum_b D_{-b,-q}^{(k)} T_k^{-b} \tag{13}$$

We can now use Shankar's hint (which I tried to prove, but couldn't, although it's probably something simple):

$$D_{-b,-q}^{(k)} = (-1)^{b-q}\left(D_{bq}^{(k)}\right)^* \tag{14}$$

Using this, we have

$$U\Omega_k|jm\rangle = \sum_{q,a,b,c} (-1)^q D_{aq}^{(k)} S_k^a (-1)^{b-q}\left(D_{bq}^{(k)}\right)^* T_k^{-b} D_{cm}^{(j)}|jc\rangle \tag{15}$$

$$= \sum_{a,b,c}\sum_q \left[D_{aq}^{(k)}\left(D_{bq}^{(k)}\right)^*\right](-1)^b S_k^a T_k^{-b} D_{cm}^{(j)}|jc\rangle \tag{16}$$

Because $D_{aq}^{(k)}$ is a unitary matrix (it's the matrix elements of the unitary rotation operator $D_{aq}^{(k)} = \langle ka|U[R]|kq\rangle$) its rows are orthonormal (see Shankar, Theorem 8 in chapter 1), so the sum over q is

$$\sum_q D_{aq}^{(k)}\left(D_{bq}^{(k)}\right)^* = \delta_{ab} \tag{17}$$

Therefore, the rotated state is

$$U\Omega_{k}\left|jm\right\rangle = \sum_{a,b,c}\delta_{ab}\left(-1\right)^{b}S_{k}^{a}T_{k}^{-b}D_{cm}^{(j)}\left|jc\right\rangle \tag{18}$$

$$= \left[\sum_{b}\left(-1\right)^{b}S_{k}^{b}T_{k}^{-b}\right]\left[\sum_{c}D_{cm}^{(j)}\left|jc\right\rangle\right] \tag{19}$$

$$= \Omega_{k}U\left[R\right]\left|jm\right\rangle \tag{20}$$

In other words, the operator Ω_k is unchanged by rotation, as the same operator operates on the rotated state $U\left[R\right]\left|jm\right\rangle$. Therefore, Ω_k is a scalar for all k.

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