

SPHERICAL TENSOR OPERATORS; A SCALAR OPERATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.2.

A spherical tensor operator is defined by the way it transforms under rotation:

$$U[R]T_k^qU^\dagger[R] = \sum_{q'} D_{q'q}^{(k)} T_k^{q'} \quad (1)$$

where $D^{(k)}$ is the k -th block in the block diagonal matrix formed from the angular momentum operators J . We can form an operator Ω from two spherical tensor operators:

$$\Omega_k \equiv \sum_q (-1)^q S_k^q T_k^{-q} \quad (2)$$

For $k = 1$ we can write a spherical tensor operator in terms of a 3-d vector operator. We'll use lower-case letters to represent the vector operator, so we have

$$S_1^{\pm 1} = \mp \frac{s_x \pm is_y}{\sqrt{2}} \quad (3)$$

$$S_1^0 = s_z \quad (4)$$

$$T_1^{\pm 1} = \mp \frac{t_x \pm it_y}{\sqrt{2}} \quad (5)$$

$$T_1^0 = t_z \quad (6)$$

Plugging these into 2 with $k = 1$ we have

$$\Omega_1 = \frac{1}{2} (s_x + is_y)(t_x - it_y) + s_z t_z + \frac{1}{2} (s_x - is_y)(t_x + it_y) \quad (7)$$

$$= s_x t_x + s_y t_y + s_z t_z \quad (8)$$

$$= \mathbf{s} \cdot \mathbf{t} \quad (9)$$

Thus Ω_1 is the scalar product of the two vectors, and is therefore a scalar operator.

To prove this for any k , we can let Ω_k operate on an angular momentum eigenket $|jm\rangle$ and then rotate this state, using 1. To simplify things, I'll write the unitary rotation operator $U[R]$ without the explicit R dependence, so it's just U .

$$U\Omega_k|jm\rangle = U\sum_q(-1)^q S_k^q T_k^{-q}|jm\rangle \quad (10)$$

$$= \sum_q(-1)^q U S_k^q U^\dagger U T_k^{-q} U^\dagger U|jm\rangle \quad (11)$$

$$= \sum_{q,a,b,c}(-1)^q D_{aq}^{(k)} S_k^a D_{-b,-q}^{(k)} T_k^{-b} D_{cm}^{(j)}|jc\rangle \quad (12)$$

Each of the lower indices in $D_{m,m'}^{(k)}$ can take values $-k, \dots, +k$, so a sum over m is the same as a sum over $-m$. That is

$$U T_k^{-q} U^\dagger = \sum_b D_{b,-q}^{(k)} T_k^b = \sum_b D_{-b,-q}^{(k)} T_k^{-b} \quad (13)$$

We can now use Shankar's hint (which I tried to prove, but couldn't, although it's probably something simple):

$$D_{-b,-q}^{(k)} = (-1)^{b-q} \left(D_{bq}^{(k)} \right)^* \quad (14)$$

Using this, we have

$$U\Omega_k|jm\rangle = \sum_{q,a,b,c}(-1)^q D_{aq}^{(k)} S_k^a (-1)^{b-q} \left(D_{bq}^{(k)} \right)^* T_k^{-b} D_{cm}^{(j)}|jc\rangle \quad (15)$$

$$= \sum_{a,b,c} \sum_q \left[D_{aq}^{(k)} \left(D_{bq}^{(k)} \right)^* \right] (-1)^b S_k^a T_k^{-b} D_{cm}^{(j)}|jc\rangle \quad (16)$$

Because $D_{aq}^{(k)}$ is a unitary matrix (it's the matrix elements of the unitary rotation operator $D_{aq}^{(k)} = \langle ka|U[R]|kq\rangle$) its rows are orthonormal (see Shankar, Theorem 8 in chapter 1), so the sum over q is

$$\sum_q D_{aq}^{(k)} \left(D_{bq}^{(k)} \right)^* = \delta_{ab} \quad (17)$$

Therefore, the rotated state is

$$U\Omega_k|jm\rangle = \sum_{a,b,c} \delta_{ab} (-1)^b S_k^a T_k^{-b} D_{cm}^{(j)} |jc\rangle \quad (18)$$

$$= \left[\sum_b (-1)^b S_k^b T_k^{-b} \right] \left[\sum_c D_{cm}^{(j)} |jc\rangle \right] \quad (19)$$

$$= \Omega_k U[R] |jm\rangle \quad (20)$$

In other words, the operator Ω_k is unchanged by rotation, as the same operator operates on the rotated state $U[R]|jm\rangle$. Therefore, Ω_k is a scalar for all k .

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