WIGNER-ECKART THEOREM - EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.3.

The Wigner-Eckart theorem says that for any spherical tensor operator T_1^q we can write its matrix elements in the basis of angular momentum eigenstates $|\alpha l m\rangle$ as a product of two factors:

$$\left\langle \alpha_2 j_2 m_2 \left| T_k^q \right| \alpha_1 j_1 m_1 \right\rangle = \left\langle \alpha_2 j_2 \left|\left| T_k \right|\right| \alpha_1 j_1 \right\rangle \left\langle j_2 m_2 \left| kq, j_1 m_1 \right. \right\rangle \tag{1}$$

where the first factor on the RHS is the reduced matrix element, and is independent of m_1, m_2 and the tensor index q. Our earlier example went through the calculation for the position operator R_1^q, and this involved integrals over spatial coordinates. The theorem also applies to cases where the matrix elements depend only on angular momentum parameters.

First, we'll look at the rank-1 tensor J_1^q which represents total angular momentum. The tensor components are

$$J_1^{\pm 1} = \mp \frac{J_x \pm i J_y}{\sqrt{2}} = \mp \frac{J_\pm}{\sqrt{2}} \tag{2}$$

$$J_1^0 = J_z \tag{3}$$

where $J_\pm$ are the usual raising and lowering operators.

According to 1, we can write the matrix elements as

$$\left\langle \alpha_2 j_2 m_2 \left| J_1^q \right| \alpha_1 j_1 m_1 \right\rangle = \left\langle \alpha_2 j_2 \left|\left| J_1 \right|\right| \alpha_1 j_1 \right\rangle \left\langle j_2 m_2 \left| 1q, j_1 m_1 \right. \right\rangle \tag{4}$$

Since the factor $\langle \alpha_2 l_2 || J_1 || \alpha_1 l_1 \rangle$ does not depend on q, the equation must be true for the case $q = 0$, so the LHS becomes

$$\left\langle \alpha_2 j_2 m_2 \left| J_1^0 \right| \alpha_1 j_1 m_1 \right\rangle = \left\langle \alpha_2 j_2 m_2 \left| J_z \right| \alpha_1 j_1 m_1 \right\rangle \tag{5}$$

$$= m_1 \hbar \left\langle \alpha_2 j_2 m_2 \left| \alpha_1 j_1 m_1 \right. \right\rangle \tag{6}$$

$$= m_1 \hbar \delta_{\alpha_2 \alpha_1} \delta_{j_2 j_1} \delta_{m_2 m_1} \tag{7}$$

where the δs arise because the kets are orthonormal. Now suppose that we take $m_1 = m_2 = j_1 = j$, and we use the hint given by Shankar that

$$\langle jj\,|jj,10\rangle = \sqrt{\frac{j}{j+1}} \tag{8}$$

Then, from 4 we have

$$\langle\alpha_2 j_2\,||J_1||\,\alpha_1 j_1\rangle = \frac{\left\langle\alpha_2 j_2 j\left|J_1^0\right|\alpha_1 jj\right\rangle}{\langle jj\,|jj,10\rangle} \tag{9}$$

$$= \hbar j\delta_{\alpha_2\alpha_1}\delta_{j_2 j}\sqrt{\frac{j+1}{j}} \tag{10}$$

$$= \sqrt{j(j+1)}\hbar\delta_{\alpha_2\alpha_1}\delta_{j_2 j} \tag{11}$$

Now suppose we consider a more general case where the tensor operator is $\mathbf{J}\cdot\mathbf{A}$, where $\mathbf{A}$ is some arbitrary vector. We've seen earlier that the scalar product of two vectors can be written as

$$\mathbf{J}\cdot\mathbf{A} = \sum_{q=\pm1,0}(-1)^q J_1^q A_1^{-q} \tag{12}$$

where the tensors in the sum on the RHS are formed the same way as in 2. Writing out this sum gives the hint in Shankar's question, which is that we can write the scalar product as

$$\mathbf{J}\cdot\mathbf{A} = J_z A_z + \frac{1}{2}(J_-A_+ + J_+A_-) \tag{13}$$

In what follows, we'll also need the fact that

$$J_\pm^\dagger = J_\mp \tag{14}$$

$$J_z^\dagger = J_z \tag{15}$$

and that

$$J_\pm\,|\alpha,j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|\alpha,j,m\pm1\rangle \tag{16}$$

Now let's take the matrix element of $\mathbf{J}\cdot\mathbf{A}$, although this time things are made a bit easier since we take $j_2 = j_1 = j$, so the total angular momentum is the same in all matrix elements. We have

$$\left\langle \alpha' jm' \right| \mathbf{J}\cdot\mathbf{A} \left| \alpha jm \right\rangle = \left\langle \alpha' jm' \right| J_z A_z \left| \alpha jm \right\rangle + \tag{17}$$

$$\frac{1}{2}\left(\left\langle \alpha' jm' \right| J_- A_+ \left| \alpha jm \right\rangle + \left\langle \alpha' jm' \right| J_+ A_- \left| \alpha jm \right\rangle \right) \tag{18}$$

$$= \left\langle J_z \alpha' jm' \right| A_z \left| \alpha jm \right\rangle + \tag{19}$$

$$\frac{1}{2}\left(\left\langle J_+ \alpha' jm' \right| A_+ \left| \alpha jm \right\rangle + \left\langle J_- \alpha' jm' \right| A_- \left| \alpha jm \right\rangle \right) \tag{20}$$

$$= m'\hbar \left\langle \alpha' jm' \right| A_z \left| \alpha jm \right\rangle + \tag{21}$$

$$\frac{\hbar}{2}\left(\sqrt{(j-m')(j+m'+1)} \left\langle \alpha' j, m'+1 \right| A_+ \left| \alpha jm \right\rangle + \right. \tag{22}$$

$$\left. \sqrt{(j+m')(j-m'+1)} \left\langle \alpha' j, m'-1 \right| A_- \left| \alpha jm \right\rangle \right) \tag{23}$$

From 2, we have that $A_+ = -\sqrt{2}A_1^1$, $A_- = \sqrt{2}A_1^{-1}$ and $A_z = A_1^0$, so we have

$$\left\langle \alpha' jm' \right| \mathbf{J}\cdot\mathbf{A} \left| \alpha jm \right\rangle = m'\hbar \left\langle \alpha' jm' \right| A_1^0 \left| \alpha jm \right\rangle + \tag{24}$$

$$\frac{\hbar}{\sqrt{2}}\left(-\sqrt{(j-m')(j+m'+1)} \left\langle \alpha' j, m'+1 \right| A_1^1 \left| \alpha jm \right\rangle + \right. \tag{25}$$

$$\left. \sqrt{(j+m')(j-m'+1)} \left\langle \alpha' j, m'-1 \right| A_1^{-1} \left| \alpha jm \right\rangle \right) \tag{26}$$

However, from 1 we know that

$$\left\langle \alpha' jm' \right| A_1^q \left| \alpha jm \right\rangle = \left\langle \alpha' j \right\| A_1 \left\| \alpha j \right\rangle \left\langle jm' \middle| 1q, jm \right\rangle \tag{27}$$

where the first factor is the same for all q. Therefore, we can write 24 as

$$\left\langle \alpha' jm' \right| \mathbf{J}\cdot\mathbf{A} \left| \alpha jm \right\rangle = c \left\langle \alpha' j \right\| A_1 \left\| \alpha j \right\rangle \tag{28}$$

where

$$c = m'\hbar \left\langle jm' \left| 10, jm \right.\right\rangle + \tag{29}$$

$$\frac{\hbar}{\sqrt{2}}\left(\sqrt{(j+m')(j-m'+1)}\left\langle j, m'-1 \left| 1, -1, jm \right.\right\rangle - \right. \tag{30}$$

$$\left.\sqrt{(j-m')(j+m'+1)}\left\langle j, m'+1 \left| 11, jm \right.\right\rangle\right) \tag{31}$$

which is independent of α and α'.

To work out c explicitly, we need to find the bracket terms in its expression. We can do this by going back to 9 with $j_1 = j_2 = j$. We have

$$\left\langle jm' \left| J_1^q \right| jm \right\rangle = \left\langle j \left|\left| J_1 \right|\right| j \right\rangle \left\langle jm' \left| 1q, jm \right.\right\rangle \tag{32}$$

From 11 we have

$$\left\langle jm' \left| 1q, jm \right.\right\rangle = \frac{\left\langle jm' \left| J_1^q \right| jm \right\rangle}{\hbar\sqrt{j(j+1)}} \tag{33}$$

We can work out the matrix elements $\left\langle jm' \left| J_1^q \right| jm \right\rangle$ by using 2 and the raising and lowering operators. We get

$$\left\langle jm' \left| 10, jm \right. \right\rangle = \frac{\langle jm' | J_z | jm \rangle}{\hbar \sqrt{j(j+1)}} \tag{34}$$

$$= \frac{m\hbar}{\hbar} \frac{\langle jm' | jm \rangle}{\sqrt{j(j+1)}} \tag{35}$$

$$= \frac{m}{\sqrt{j(j+1)}} \delta_{mm'} \tag{36}$$

$$\left\langle j, m'-1 \left| 1-1, jm \right. \right\rangle = \frac{1}{\sqrt{2}} \frac{\langle j, m'-1 | J_- | jm \rangle}{\hbar \sqrt{j(j+1)}} \tag{37}$$

$$= \frac{1}{\sqrt{2}} \frac{\langle J_+ j, m'-1 | jm \rangle}{\hbar \sqrt{j(j+1)}} \tag{38}$$

$$= \frac{\sqrt{(j-m'+1)(j+m')}}{\sqrt{2}\sqrt{j(j+1)}} \left\langle jm' \left| jm \right. \right\rangle \tag{39}$$

$$= \frac{\sqrt{(j-m+1)(j+m)}}{\sqrt{2}\sqrt{j(j+1)}} \delta_{mm'} \tag{40}$$

$$\left\langle j, m'+1 \left| 11, jm \right. \right\rangle = -\frac{1}{\sqrt{2}} \frac{\langle j, m'+1 | J_+ | jm \rangle}{\hbar \sqrt{j(j+1)}} \tag{41}$$

$$= -\frac{1}{\sqrt{2}} \frac{\langle J_- j, m'+1 | jm \rangle}{\hbar \sqrt{j(j+1)}} \tag{42}$$

$$= -\frac{\sqrt{(j+m'+1)(j-m')}}{\sqrt{2}\sqrt{j(j+1)}} \left\langle jm' \left| jm \right. \right\rangle \tag{43}$$

$$= -\frac{\sqrt{(j+m+1)(j-m)}}{\sqrt{2}\sqrt{j(j+1)}} \delta_{mm'} \tag{44}$$

Putting everything together, we have

$$c = \frac{\hbar \delta_{mm'}}{\sqrt{j(j+1)}} \left[m^2 + \frac{1}{2}(j-m+1)(j+m) + \frac{1}{2}(j+m+1)(j-m) \right] \tag{45}$$

$$= \frac{\hbar \delta_{mm'}}{\sqrt{j(j+1)}} \left(j^2 + j \right) \tag{46}$$

$$= \hbar \sqrt{j(j+1)} \delta_{mm'} \tag{47}$$

We can combine these results to get an expression for the matrix elements of A_1^q. From 27, 28, 33 and 47 we have

$$\left\langle \alpha' jm' \left| A_1^q \right| \alpha jm \right\rangle = \left\langle \alpha' j \left|\left| A_1 \right|\right| \alpha j \right\rangle \left\langle jm' \left| 1q, jm \right. \right\rangle \tag{48}$$

$$= \frac{1}{c} \left\langle \alpha' jm' \left| \mathbf{J} \cdot \mathbf{A} \right| \alpha jm \right\rangle \left\langle jm' \left| 1q, jm \right. \right\rangle \tag{49}$$

$$= \frac{\left\langle \alpha' jm' \left| \mathbf{J} \cdot \mathbf{A} \right| \alpha jm \right\rangle}{\hbar^2 j (j+1)} \left\langle jm' \left| J_1^q \right| jm \right\rangle \tag{50}$$

PINGBACKS

Pingback: Wigner-Eckart Theorem - adding orbital and spin angular momenta