WIGNER-ECKART THEOREM - ADDING ORBITAL AND SPIN ANGULAR MOMENTA

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 15.3; Exercise 15.3.4 - 15.3.5.

The Wigner-Eckart theorem says that for any spherical tensor operator T_1^q we can write its matrix elements in the basis of angular momentum eigenstates $|\alpha lm\rangle$ as a product of two factors:

$$\left\langle \alpha_2 j_2 m_2 \left| T_k^q \right| \alpha_1 j_1 m_1 \right\rangle = \left\langle \alpha_2 j_2 \left|\left| T_k \right|\right| \alpha_1 j_1 \right\rangle \left\langle j_2 m_2 \left| kq, j_1 m_1 \right.\right\rangle \tag{1}$$

where the first factor on the RHS is the reduced matrix element, and is independent of m_1, m_2 and the tensor index q.

We can apply this to the case where we have a particle such as a proton or electron with both orbital and spin angular momentum. Such a particle has a magnetic moment

$$\mu = \gamma_1 \mathbf{J}_1 + \gamma_2 \mathbf{J}_2 \tag{2}$$

Here γ_i are the gyromagnetic ratios for the two angular momenta. Suppose we want to find $\langle \mu \rangle$ for a particle in a state $|jm, j_1 j_2\rangle$, where this ket represents a state with the two component momenta j_1 and j_2 and total angular momentum j with z component m. [This is different from the notation that Shankar uses in most of Chapter 15, where $|j_1 m_1, j_2 m_2\rangle$ represents a state with the two components j_1 and j_2 and their corresponding z components.] We can use the formula derived earlier

$$\left\langle \alpha' jm' \left| A_1^q \right| \alpha jm \right\rangle = \frac{\left\langle \alpha' jm' \left| \mathbf{J} \cdot \mathbf{A} \right| \alpha jm \right\rangle}{\hbar^2 j (j+1)} \left\langle jm' \left| J_1^q \right| jm \right\rangle \tag{3}$$

We can work out 2 by applying this formula to each momentum component separately. As we're concerned only with angular momentum we can omit α (since it represents other parameters) and set $m' = m$. The notation can be a bit confusing, since in this problem, the subscript 1 or 2 represents a label for a particular angular momentum and not the rank of a tensor. With $\mathbf{A} = \mathbf{J}_1$ we have

$$\langle jm, j_1 j_2 | J_{1i} | jm, j_1 j_2 \rangle = \frac{\langle jm, j_1 j_2 | \mathbf{J} \cdot \mathbf{J}_1 | jm, j_1 j_2 \rangle}{\hbar^2 j(j+1)} \langle jm, j_1 j_2 | J_i | jm, j_1 j_2 \rangle \tag{4}$$

The subscript i represents the component x, y or z. We can workout the middle matrix element using

$$\mathbf{J} \cdot \mathbf{J}_1 = J_1^2 + \mathbf{J}_2 \cdot \mathbf{J}_1 \tag{5}$$

$$= J_1^2 + \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right) \tag{6}$$

$$= \frac{1}{2}\left(J^2 + J_1^2 - J_2^2\right) \tag{7}$$

Therefore

$$\langle jm, j_1 j_2 | \mathbf{J} \cdot \mathbf{J}_1 | jm, j_1 j_2 \rangle = \frac{\hbar^2}{2}\left(j(j+1) + j_1(j_1+1) - j_2(j_2+1)\right) \tag{8}$$

Applying the same calculation for $\mathbf{J} \cdot \mathbf{J}_2$ we have

$$\langle jm, j_1 j_2 | \mathbf{J} \cdot \mathbf{J}_2 | jm, j_1 j_2 \rangle = \frac{\hbar^2}{2}\left(j(j+1) + j_2(j_2+1) - j_1(j_1+1)\right) \tag{9}$$

We also have

$$\langle jm, j_1 j_2 | J_z | jm, j_1 j_2 \rangle = m\hbar \tag{10}$$

From the raising and lowering operators we have

$$J_x = \frac{1}{2}\left(J_+ + J_-\right) \tag{11}$$

$$J_y = \frac{1}{2i}\left(J_+ - J_-\right) \tag{12}$$

Because kets with different m values are orthogonal and $J_\pm$ raises or lowers the m value, we have

$$\langle jm, j_1 j_2 | J_\pm | jm, j_1 j_2 \rangle = 0 \tag{13}$$

Therefore

$$\langle jm, j_1 j_2 | J_{x,y} | jm, j_1 j_2 \rangle = 0 \tag{14}$$

Putting these into 2 gives

$$\langle \mu_x \rangle = \langle \mu_y \rangle = 0 \tag{15}$$

$$\langle \mu_z \rangle = \frac{\hbar^2}{2\hbar^2 j(j+1)} \left[\gamma_1 (j(j+1) + j_1(j_1+1) - j_2(j_2+1)) + \right. \tag{16}$$

$$\left. \gamma_2 (j(j+1) + j_2(j_2+1) - j_1(j_1+1)) \right] \hbar m \tag{17}$$

$$= \frac{\hbar m}{2} \left[\gamma_1 + \gamma_2 + (\gamma_1 - \gamma_2) \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \right] \tag{18}$$

For a proton in the state ${}^2P_{1/2}$, the orbital angular momentum is $j_1 = 1$ (from the P), the spin is $j_2 = \frac{1}{2}$ (from $2S+1 = 2$) and the total angular momentum is $j = \frac{1}{2}$ (from the subscript). The orbital gyromagnetic ratio is (Shankar, eqn 14.4.7)

$$\gamma_1 = \frac{e}{2Mc} \tag{19}$$

The spin gyromagnetic ratio is (Shankar, p. 391):

$$\gamma_2 = 5.6 \frac{e}{2Mc} \tag{20}$$

Plugging these into 18 we get

$$\langle \mu_z \rangle = 0.53 m \frac{e\hbar}{2Mc} \tag{21}$$

The z component of total angular momentum can take values of $\pm\frac{1}{2}$ for $j = \frac{1}{2}$, so we have

$$\langle \mu_z \rangle = \pm 0.267 \frac{e\hbar}{2Mc} \tag{22}$$

where $\frac{e\hbar}{2Mc}$ is the nuclear Bohr magneton.

For an electron in the state ${}^2P_{1/2}$ everything is the same except that the spin gyromagnetic ratios are

$$\gamma_1 = -\frac{e}{2m_e c} \tag{23}$$

$$\gamma_2 = -\frac{e}{m_e c} \tag{24}$$

Plugging these into 18 gives

$$\langle \mu_z \rangle = \pm \frac{1}{3} \frac{e\hbar}{2m_e c} \tag{25}$$

where $\frac{e\hbar}{2m_e c}$ is electron Bohr magneton.

Finally, we can note a condition of the matrix elements of a general spherical tensor T_k^q, which we can see from 1. If $j_1 = j_2 = j$ and $m_1 = m_2 = m$ we have (disregarding α):

$$\left\langle jm \left| T_k^q \right| jm \right\rangle = \langle j \| T_k \| j \rangle \langle jm | kq, jm \rangle \tag{26}$$

The factor $\langle jm | kq, jm \rangle$ is a Clebsch-Gordan coefficient (up to a numerical factor), which means that it must be possible to form the angular momentum in the bra by adding the two angular momenta in the ket. This gives the condition that

$$|k - j| \leq j \leq k + j \tag{27}$$

If $k > j$ this gives the condition

$$0 \leq k \leq 2j \tag{28}$$

Any values of $k > 2j$ give a zero Clebsch-Gordan coefficient, so $\left\langle jm \left| T_k^q \right| jm \right\rangle = 0$ for $k > 2j$.