Here we’ll apply the variational principle again to the harmonic oscillator, this time with potential

\[ \psi(x) = \begin{cases} 
(x - a)^2 (x + a)^2 & x \leq |a| \\
0 & |x| > a 
\end{cases} \quad (1) \]

Here \( a \) is the parameter to be varied, and we can see that it controls the width of the trial wave function as well as its height. We first find the normalization constant

\[ N \equiv \langle \psi | \psi \rangle = \int_{-a}^{a} (x - a)^4 (x + a)^4 \, dx \quad (2) \]

\[ = \frac{256}{315} a^9 \quad (3) \]

where I used Maple to do and simplify the integral. If you want to do it by hand, it’s probably easiest to use the substitution \( u = x - a \) before multiplying out the factors in the integrand.

The energy estimate is then obtained by minimizing

\[ E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \quad (4) \]

where the Hamiltonian contains the harmonic oscillator potential:

\[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \quad (5) \]

To calculate \( \langle \psi | H | \psi \rangle \) requires integrating a sixth-degree polynomial which is straightforward but very tedious to do by hand if you like (which is probably why the exercise is marked as ‘optional’ in Shankar), but again I used Maple to get...
\[ \frac{d^2 \psi}{dx^2} = 12x^2 - 4a^2 \]  

\[ \langle \psi | H | \psi \rangle = \int_{-a}^{a} (x-a)^2 (x+a)^2 \left[ \frac{\hbar^2}{2m} (12x^2 - 4a^2) + \frac{1}{2} m \omega^2 x^2 (x-a)^2 (x+a)^2 \right] dx \]

\[ = \frac{128}{3465} \left( a^{11} m \omega^2 + 33 \frac{\hbar^2 a^7}{m} \right) \]

The expression to minimize is therefore

\[ E = \frac{128}{3465} \left( a^{11} m \omega^2 + 33 \frac{\hbar^2 a^7}{m} \right) \times \frac{315}{256a^9} \]

\[ = \frac{1}{22} \left( a^2 m \omega^2 + 33 \frac{\hbar^2}{m} a^{-2} \right) \]

Taking the derivative, we need to solve

\[ \frac{dE}{da} = \frac{1}{11} \left( a m \omega^2 - 33 \frac{\hbar^2}{m} a^{-3} \right) = 0 \]

This gives an optimum value for \( a \):

\[ a_0 = 33^{1/4} \sqrt{\frac{\hbar}{m \omega}} \]

Substituting into 11, we get the estimate of the ground state energy

\[ E_0 = \frac{\sqrt{33}}{11} \hbar \omega \simeq 0.522 \hbar \omega \]

The exact ground state energy for the harmonic oscillator is \( \frac{1}{2} \hbar \omega \) so this estimate is reasonably good.

This answer agrees with the back-of-the-book answer in Shankar, since

\[ \frac{\sqrt{33}}{11} = \frac{1}{2} \sqrt{\frac{4 \times 33}{11}} = \frac{1}{2} \sqrt{\frac{12 \times 11}{11}} = \frac{1}{2} \sqrt{\frac{12}{11}} \]