ROTATION ABOUT AN ARBITRARY AXIS IN 3-D

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Post date: 15 Mar 2018.

References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2.

In discussing Lorentz transformations, we pointed out that a general infinitesimal Lorentz transformation has 6 independent parameters, of which 3 represent a general rotation in 3-d space. Such an infinitesimal transformation has the form

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \delta\omega^{\mu}_{\nu} \tag{1}$$

Srednicki then states that a general infinitesimal rotation by an angle $\delta\alpha$ about an arbitrary 3-d axis given by the unit vector

$$\hat{\mathbf{n}} = [n_x, n_y, n_z] \tag{2}$$

is given by

$$\delta\omega_{ij} = -\varepsilon_{ijk} n_k \delta\alpha \tag{3}$$

It's an interesting exercise to see where this formula comes from.

First, we can write the components of $\hat{\mathbf{n}}$ in terms of the usual spherical angles ϕ and θ:

$$n_x = \sin\theta\cos\phi \tag{4}$$
$$n_y = \sin\theta\sin\phi \tag{5}$$
$$n_z = \cos\theta \tag{6}$$

To rotate a 3-d point about $\hat{\mathbf{n}}$ by $\delta\alpha$, we can first rotate $\hat{\mathbf{n}}$ so it lies in the xz plane, then rotate it again so that it lies along the z axis. We can then apply the rotation by $\delta\alpha$ about the z axis. We then invert the first two rotations to rotate $\hat{\mathbf{n}}$ back to its original positions. Since we know the rotation matrices about each axis separately, the problem reduces to a matrix multiplication.

The rotation of $\hat{\mathbf{n}}$ into the xz plane requires a rotation by $-\phi$ about the z axis. The matrix for this is

$$T_{xz}=\left[\begin{array}{ccc}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{array}\right] \tag{7}$$

The rotation from here into the z axis is a rotation by $-\theta$ about the y axis, for which the matrix is

$$T_{z}=\left[\begin{array}{ccc}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{array}\right] \tag{8}$$

Now the rotation by a general angle α about the z axis is

$$R(\alpha)=\left[\begin{array}{ccc}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{array}\right] \tag{9}$$

To reverse the first two rotations we need the inverses of T_{xz} and T_z, which we can obtain by just swapping the off-diagonal entries:

$$T_{xz}^{-1}=\left[\begin{array}{ccc}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{array}\right] \tag{10}$$

$$T_{z}^{-1}=\left[\begin{array}{ccc}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{array}\right] \tag{11}$$

Thus the net effect of rotating by α about $\hat{\mathbf{n}}$ is given by

$$R(\alpha\hat{\mathbf{n}})=T_{xz}^{-1}T_{z}^{-1}R(\alpha)\,T_{z}T_{xz} \tag{12}$$

The full expression for a finite angle α is quite messy, but for an infinitesimal rotation by $\delta\alpha$, we can replace $\cos(\delta\alpha)$ by 1 and $\sin(\delta\alpha)$ by $\delta\alpha$, which simplifies things a lot. The final result is

$$R(\alpha\hat{\mathbf{n}})=\left[\begin{array}{ccc}1 & -\cos\theta\delta\alpha & \sin\theta\sin\phi\delta\alpha\\ \cos\theta\delta\alpha & 1 & \sin\theta\cos\phi\delta\alpha\\ -\sin\theta\sin\phi\delta\alpha & -\sin\theta\cos\phi\delta\alpha & 1\end{array}\right] \tag{13}$$

In terms of the components of $\hat{\mathbf{n}}$, given by 4, this is

$$R(\alpha\hat{\mathbf{n}}) = \begin{bmatrix} 1 & -n_z\delta\alpha & n_y\delta\alpha \\ n_z\delta\alpha & 1 & n_x\delta\alpha \\ -n_y\delta\alpha & -n_x\delta\alpha & 1 \end{bmatrix} \tag{14}$$

$$= I + \delta\alpha \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & n_x \\ -n_y & -n_x & 0 \end{bmatrix} \tag{15}$$

Comparing this with 3, we see that the incremental components are indeed given by 3.