DIRAC EQUATION

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 1, Problem 1.1.

The Klein-Gordon equation is an early attempt at a relativistic quantum theory, but it contains a second-order time derivative which leads to probability not being conserved over time. Dirac proposed another equation that attempts to solve this problem for particles of spin 1/2. The Dirac equation is essentially a modification of the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi_a(x)=\left[-i\hbar c\left(\alpha^j\right)_{ab}\partial_j+mc^2(\beta)_{ab}\right]\psi_b(x) \tag{1}$$

Here, ψ is now a vector in spin space with components ψ_a. The objects β and α^j (for $j=1,2,3$) are square matrices (where the subscript ab indicates the component of the matrix being considered), also in spin space, and repeated indices are summed over spatial coordinates only. [We won't worry about *how* Dirac arrived at this equation for now; we'll just accept it and see where it leads.]

To make this equation formally equivalent to the Schrödinger equation, the hamiltonian operator H on the RHS must now be a matrix. We can also use the definition of the momentum operator $P_j=-i\hbar\partial_j$ to get

$$H_{ab}=cP_j\left(\alpha^j\right)_{ab}+mc^2(\beta)_{ab} \tag{2}$$

This might not look much like the relativistic energy:

$$E=\sqrt{p^2c^2+m^2c^4} \tag{3}$$

but if we square 2 (remembering that matrix products need not commute), we have

$$\left(H^2\right)_{ab}=c^2P_jP_k\left(\alpha^j\alpha^k\right)_{ab}+mc^3P_j\left(\alpha^j\beta+\beta\alpha^j\right)_{ab}+m^2c^4\left(\beta^2\right)_{ab} \tag{4}$$

We can define the anticommutator as

$$\{A,B\}\equiv AB+BA \tag{5}$$

We can write the first term on the RHS of 4 as

$$c^2P_jP_k\left(\alpha^j\alpha^k\right)_{ab}=\frac{1}{2}c^2P_jP_k\left\{\alpha^j,\alpha^k\right\}_{ab} \tag{6}$$

so we get

$$\left(H^2\right)_{ab}=\frac{1}{2}c^2P_jP_k\left\{\alpha^j,\alpha^k\right\}_{ab}+mc^3P_j\left\{\alpha^j,\beta\right\}_{ab}+m^2c^4\left(\beta^2\right)_{ab} \tag{7}$$

In order to make this equal to E^2, we need the matrices α^j and β to satisfy the conditions:

$$\left\{\alpha^j,\alpha^k\right\}_{ab} = 2\delta^{jk}\delta_{ab} \tag{8}$$

$$\left\{\alpha^j,\beta\right\} = 0 \tag{9}$$

$$\left(\beta^2\right)_{ab} = \delta_{ab} \tag{10}$$

The first condition requires the anticommutator of α^j and α^k to be zero unless $j=k$, in which case the anticommutator gives the identity matrix. Remember that the superscripts j and k specify which *matrix* we're talking about, while the subscripts ab indicate the *component* of the matrix. The conditions aren't derived; rather they are imposed to make the energy come out right. With these conditions, we have

$$\left(H^2\right)_{ab}=\left(\mathbf{P}^2c^2+m^2c^4\right)\delta_{ab} \tag{11}$$

which gives the correct operator for the square of the energy.

The question arises as to what these matrices α^j and β are. One candidate is the set of three Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{13}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{14}$$

By direct calculation, we see that $\left\{\sigma^i,\sigma^j\right\}=2\delta^{ij}$. For example

$$\{\sigma_x,\sigma_y\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{15}$$
$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{16}$$
$$= 0 \tag{17}$$
$$\{\sigma_x,\sigma_x\} = 2\sigma_x^2 \tag{18}$$
$$= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{19}$$

and so on. However, in order to satisfy 9, we need to find a single matrix that anticommutes with all 3 spin matrices. We get

$$\{\sigma_x,\beta\} = \begin{bmatrix} \beta_{21} & \beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix} + \begin{bmatrix} \beta_{12} & \beta_{11} \\ \beta_{22} & \beta_{21} \end{bmatrix} \tag{20}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

This gives

$$\beta_{12} = -\beta_{21} \equiv \gamma \tag{22}$$
$$\beta_{11} = -\beta_{22} \equiv \varepsilon \tag{23}$$

We then get

$$\{\sigma_z,\beta\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{bmatrix} + \begin{bmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{24}$$
$$= \begin{bmatrix} \varepsilon & \gamma \\ \gamma & \varepsilon \end{bmatrix} + \begin{bmatrix} \varepsilon & -\gamma \\ -\gamma & \varepsilon \end{bmatrix} \tag{25}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

This gives

$$\varepsilon = 0 \tag{27}$$

So finally

$$\{\sigma_y, \beta\} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{28}$$

$$= \begin{bmatrix} i\gamma & 0 \\ 0 & i\gamma \end{bmatrix} + \begin{bmatrix} i\gamma & 0 \\ 0 & i\gamma \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

So $\gamma = 0$ resulting in $(\beta)_{ab} = 0$. Thus there is no non-zero matrix β that anticommutes with all 3 of the Pauli spin matrices.

So what *can* we say about the Dirac matrices? From 10, we see that the eigenvalues of $\beta^2 = I$ are all 1, so the eigenvalues of β must be ± 1.

The trace (sum of the diagonal elements) of a matrix is equal to the sum of its eigenvalues (theorem from matrix algebra). To find the trace of β, we can use the anticommutators 8 and 9, together with another theorem from matrix algebra which states that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any square matrices A and B of the same order.

$$\mathrm{tr}\left(\alpha_1^2 \beta\right) = \mathrm{tr}\left(\alpha_1\left(\alpha_1 \beta\right)\right) \tag{31}$$

$$= \mathrm{tr}\left(\left(\alpha_1 \beta\right) \alpha_1\right) \tag{32}$$

However, from 9, $\alpha_1 \beta = -\beta \alpha_1$ and from 8, $\alpha_1^2 = I$ (the identity matrix), so

$$\mathrm{tr}\left(\alpha_1^2 \beta\right) = \mathrm{tr}(\beta) \tag{33}$$

$$= \mathrm{tr}\left(\left(\alpha_1 \beta\right) \alpha_1\right) \tag{34}$$

$$= -\mathrm{tr}\left(\left(\beta \alpha_1\right) \alpha_1\right) \tag{35}$$

$$= -\mathrm{tr}\left(\beta \alpha_1^2\right) \tag{36}$$

$$= -\mathrm{tr}(\beta) \tag{37}$$

Hence $\mathrm{tr}(\beta) = -\mathrm{tr}(\beta) = 0$, so β must have an equal number of $+1$ and -1 eigenvalues. In other words, β must be even dimensional, so the smallest size is 4×4.

We can also find the trace of α^j by starting with $\mathrm{tr}\left(\alpha^j \beta^2\right)$ and following through the same steps as above (using $\beta^2 = I$) to show that $\mathrm{tr}\left(\alpha^j\right) = -\mathrm{tr}\left(\alpha^j\right) = 0$.

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