QUANTUM FIELD THEORY REPRESENTATION OF NON-RELATIVISTIC QUANTUM MECHANICS

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 1, Problem 1.2.

One of the main problems faced in developing a relativistic quantum theory is that in non-relativistic quantum mechanics, position and time are not on an equal footing. Position is treated as an operator, while time is just a parameter that labels a particular instance of a state or wave function. In special relativity, space and time are treated equivalently, in the sense that they form equal components of four-dimensional spacetime. [Time and space aren't the *same*, of course, since although we can travel to any point in space whenever we want, we can move only forward in time, even in relativity. However, space and time components are *treated* equally in the sense that they transform into each other in the Lorentz transformations.]

Attempts to develop a relativistic quantum theory therefore can take one of two paths in an attempt to solve this disparity. One way is to promote time to an operator, but this leads to complex theories (although they do work). The other way is to demote position from an operator to just a label, so its status is the same as that of time. This idea leads to quantum field theory.

The idea is that the position $\mathbf{x}$ becomes, like time, a label on an operator. We can define a set of operators $\phi(\mathbf{x})$ such that at each point $\mathbf{x}$ in space, there is a separate operator. The position $\mathbf{x}$ becomes a label telling us which operator we're dealing with. The set of all such operators (that is, the set of operators defined over all space) is called a *quantum field*, and hence we get quantum field theory by studying such sets of operators. In the general case, each operator is also a function of time so that a quantum field is actually made up of a set of operators $\phi(\mathbf{x},t)$.

To get an idea of how this works, we can rewrite non-relativistic quantum mechanics using a quantum field. Non-relativistic quantum mechanics is governed by the Schrödinger equation, which in its most general form is

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \tag{1}$$

where H is the hamiltonian. Now suppose we define a quantum field $a(\mathbf{x})$ and its hermitian conjugate $a^\dagger(\mathbf{x})$ that satisfy the commutation relations

$$\left[a(\mathbf{x}), a\left(\mathbf{x}'\right)\right] = 0 \tag{2}$$

$$\left[a^\dagger(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] = 0 \tag{3}$$

$$\left[a(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] = \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{4}$$

where $\delta^3(\mathbf{x})$ is the 3-d Dirac delta function. These operators are similar to the raising and lowering operators we used to solve the harmonic oscillator, although in this case the operators are labelled by positions in space rather than energy states in an oscillator.

We require one additional property for this field: if $|0\rangle$ represents the *vacuum state*, that is, a state with no particles in it, then

$$a(\mathbf{x})|0\rangle = 0 \tag{5}$$

That is, $a(\mathbf{x})$ eliminates the vacuum state for all values of $\mathbf{x}$.

Now suppose we define a hamiltonian using this field, as follows:

$$H = \int d^3x\, a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) + \frac{1}{2}\int d^3x\, d^3y\, V(\mathbf{x}-\mathbf{y})\, a^\dagger(\mathbf{x})\, a^\dagger(\mathbf{y})\, a(\mathbf{y})\, a(\mathbf{x}) \tag{6}$$

Here, U is an external potential energy and V is an interaction energy between two particles at locations $\mathbf{x}$ and $\mathbf{y}$.

Also, suppose we have a time-dependent quantum state defined by

$$|\psi,t\rangle = \int d^3x_1 \ldots d^3x_n \psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\, a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle \tag{7}$$

If we require 1 to be true, what condition does this place on the function $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)$ inside the integral? To determine this, we need to apply H to $|\psi,t\rangle$ by using the commutation relations. Consider the first integral in 6. We can propagate the operator $a(\mathbf{x})$ through the list of operators $a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)$ in 7 by applying the commutator 4. We get

(8)

$$a(\mathbf{x})\,a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n) = \left[\delta^3(\mathbf{x}-\mathbf{x}_1) + a^\dagger(\mathbf{x}_1)\,a(\mathbf{x})\right] a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_n)$$

$$= \delta^3(\mathbf{x}-\mathbf{x}_1)\,a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_n) +$$

(9)
$$a^\dagger(\mathbf{x}_1)\left[\delta^3(\mathbf{x}-\mathbf{x}_2) + a^\dagger(\mathbf{x}_2)\,a(\mathbf{x})\right] a^\dagger(\mathbf{x}_3)\ldots a^\dagger(\mathbf{x}_n)$$

$$= \delta^3(\mathbf{x}-\mathbf{x}_1)\,a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_n) + \ldots +$$

(10)
$$a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_{n-1})\left[\delta^3(\mathbf{x}-\mathbf{x}_n) + a^\dagger(\mathbf{x}_n)\,a(\mathbf{x})\right]$$

When we apply this expansion to the vacuum state $|0\rangle$, the last term in the final bracket vanishes because of 5. Doing the integral over x in 6 results in

$$\int d^3x\, a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x})\,|\psi,t\rangle =$$

(11)

$$\sum_i \int d^3x_1\ldots d^3x_n a^\dagger(\mathbf{x}_i)\left(-\frac{\hbar^2}{2m}\nabla_i^2 + U(\mathbf{x}_i)\right)\psi a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_{i-1})\,a^\dagger(\mathbf{x}_{i+1})\ldots a^\dagger(\mathbf{x}_n)\,|0\rangle =$$

(12)
$$\sum_i \int d^3x_1\ldots d^3x_n\left(-\frac{\hbar^2}{2m}\nabla_i^2 + U(\mathbf{x}_i)\right)\psi a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)$$

where the last line follows from 3.

We can do a similar calculation for the second integral in 6, although it's a bit more complicated because we have to integrate over both x and y. Applying $a(\mathbf{x})$ to $a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)$ gives the result 10. Applying $a(\mathbf{y})$ to the first term of this result gives

$$a(\mathbf{y})\,\delta^3(\mathbf{x}-\mathbf{x}_1)\,a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_n)\,|0\rangle = \delta^3(\mathbf{x}-\mathbf{x}_1)\times$$

(13)
$$\left[\delta^3(\mathbf{y}-\mathbf{x}_2)\,a^\dagger(\mathbf{x}_3)\ldots a^\dagger(\mathbf{x}_n) + \delta^3(\mathbf{y}-\mathbf{x}_n)\,a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_{n-1})\right]|0$$

with similar terms arising from the other terms in 10. When we integrate over x and y this first term gives us

(14)
$$\sum_{j=2}^{n}\int d^3x_1\ldots d^3x_n V(\mathbf{x}_1-\mathbf{x}_j)\,\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)\,|0\rangle$$

The other terms in the expansion 10 each contribute a sum

(15)
$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \int d^3x_1 \ldots d^3x_n V\left(\mathbf{x}_i - \mathbf{x}_j\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle$$

so the overall result for the second integral is

$$\frac{1}{2} \int d^3x\, d^3y\, V\left(\mathbf{x}-\mathbf{y}\right) a^\dagger\left(\mathbf{x}\right) a^\dagger\left(\mathbf{y}\right) a\left(\mathbf{y}\right) a\left(\mathbf{x}\right) |\psi, t\rangle =$$

(16)
$$\sum_{j=1}^{n} \sum_{i=1}^{j-1} \int d^3x_1 \ldots d^3x_n V\left(\mathbf{x}_i - \mathbf{x}_j\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle$$

Requiring 12 plus 16 to satisfy 1 gives us (from equating the integrands on both sides):

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{i=1}^{n} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + U\left(\mathbf{x}_i\right) \right) \psi + \sum_{j=1}^{n} \sum_{i=1}^{j-1} V\left(\mathbf{x}_i - \mathbf{x}_j\right) \psi \tag{17}$$

This is just the Schrödinger equation in its more traditional form, for a hamiltonian containing kinetic energy, overall potential energy U and particle-particle interaction energy V for a collection of n particles. Thus the state $a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle$ corresponds to a state containing n particles at locations $\mathbf{x}_i$, so the hermitian conjugate field operators $a^\dagger\left(\mathbf{x}_i\right)$ act as creation operators, with each operator creating its particle at the location $\mathbf{x}_i$ used to label the operator.

The operator

$$N \equiv \int d^3x\, a^\dagger\left(\mathbf{x}\right) a\left(\mathbf{x}\right) \tag{18}$$

counts the particles present, in effect by annihilating a particle at location $\mathbf{x}$ (if one exists there), then creating it again in the same location. More formally, if we apply N to the state $a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle$ then the annihilation operator $a\left(\mathbf{x}\right)$ acting on the state produces the result 10, so we get

$$Na^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle = \int d^3x\, a^\dagger(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{x}_1)\,a^\dagger(\mathbf{x}_2)\ldots a^\dagger(\mathbf{x}_n)|0\rangle + \ldots +$$

$$\int d^3x\, a^\dagger(\mathbf{x})\,a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_{n-1})\,\delta^3(\mathbf{x}-\mathbf{x}_n)|0\rangle \tag{19}$$

$$= \sum_{i=1}^{n} a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle \tag{20}$$

$$= n\,a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle \tag{21}$$

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