

NUMBER OPERATOR

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 1, Problem 1.3.

The number operator is defined as

$$N \equiv \int d^3x a^\dagger(\mathbf{x}) a(\mathbf{x}) \quad (1)$$

Applied to a quantum state, it counts the number of particles in that state:

$$N a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle = n a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (2)$$

Another property of N is that it commutes with any other operator that contains an equal number of creation and annihilation operators. To see this, look at the individual commutators as follows (where $a_i \equiv a(\mathbf{x}_i)$).

$$[N, a_i^\dagger] = \int d^3x \left(a^\dagger(\mathbf{x}) a(\mathbf{x}) a_i^\dagger - a_i^\dagger a^\dagger(\mathbf{x}) a(\mathbf{x}) \right) \quad (3)$$

$$= \int d^3x \left[a^\dagger(\mathbf{x}) \left(\delta(\mathbf{x} - \mathbf{x}_i) + a_i^\dagger a(\mathbf{x}) \right) - a_i^\dagger a^\dagger(\mathbf{x}) a(\mathbf{x}) \right] \quad (4)$$

$$= \int d^3x a^\dagger(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_i) \quad (5)$$

$$= a_i^\dagger \quad (6)$$

$$[N, a_i] = \int d^3x \left(a^\dagger(\mathbf{x}) a(\mathbf{x}) a_i - a_i a^\dagger(\mathbf{x}) a(\mathbf{x}) \right) \quad (7)$$

$$= \int d^3x \left[a^\dagger(\mathbf{x}) a(\mathbf{x}) a_i - \left(\delta(\mathbf{x} - \mathbf{x}_i) + a^\dagger(\mathbf{x}) a_i \right) a(\mathbf{x}) \right] \quad (8)$$

$$= - \int d^3x a(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_i) \quad (9)$$

$$= -a_i \quad (10)$$

Here we've used the commutation relations

$$[a(\mathbf{x}), a(\mathbf{x}')] = 0 \quad (11)$$

$$[a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] = 0 \quad (12)$$

$$[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}') \quad (13)$$

Now suppose we have an operator X which contains n creation operators a_i^\dagger , $i = 1, \dots, n$ and m annihilation operators a_j , $j = 1, \dots, m$:

$$X = a_{i_1}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m} \quad (14)$$

Then

$$[N, X] = Na_{i_1}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m} - a_{i_1}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m} N \quad (15)$$

$$= (a_{i_1}^\dagger N + a_{i_1}^\dagger) a_{i_2}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m} - a_{i_1}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m} N \quad (16)$$

$$= X + a_{i_1}^\dagger [N, a_{i_2}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m}] \quad (17)$$

We can see that the commutator in the last line can be worked out recursively until we've processed all the creation operators up to $a_{i_n}^\dagger$, giving

$$[N, X] = nX + a_{i_1}^\dagger \dots a_{i_n}^\dagger [N, a_{j_1} \dots a_{j_m}] \quad (18)$$

The last commutator gives us

$$[N, a_{j_1} \dots a_{j_m}] = Na_{j_1} \dots a_{j_m} - a_{j_1} \dots a_{j_m} N \quad (19)$$

$$= (a_{j_1} N - a_{j_1}) a_{j_2} \dots a_{j_m} - a_{j_1} \dots a_{j_m} N \quad (20)$$

$$= -a_{j_1} \dots a_{j_m} + a_{j_1} [N, a_{j_2} \dots a_{j_m}] \quad (21)$$

$$= -m(a_{j_1} \dots a_{j_m}) \quad (22)$$

Therefore

$$a_{i_1}^\dagger \dots a_{i_n}^\dagger [N, a_{j_1} \dots a_{j_m}] = -m(a_{i_1}^\dagger \dots a_{i_n}^\dagger a_{j_1} \dots a_{j_m}) \quad (23)$$

$$= -mX \quad (24)$$

$$[N, X] = (n - m)X \quad (25)$$

So if $n = m$ (the numbers of creation and annihilation operators are equal), the operator X commutes with N . In particular, the hamiltonian we met last time satisfies this criterion, so $[N, H] = 0$ and this hamiltonian conserves particle numbers.

PINGBACKS

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