NUMBER OPERATOR

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 1, Problem 1.3.

The number operator is defined as

$$N \equiv \int d^3x\, a^\dagger(\mathbf{x})\, a(\mathbf{x}) \tag{1}$$

Applied to a quantum state, it counts the number of particles in that state:

$$Na^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle = na^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle \tag{2}$$

Another property of N is that it commutes with any other operator that contains an equal number of creation and annihilation operators. To see this, look at the individual commutators as follows (where $a_i \equiv a(\mathbf{x}_i)$).

$$\begin{aligned}
\left[N, a_i^\dagger\right] &= \int d^3x\, \left(a^\dagger(\mathbf{x})\, a(\mathbf{x})\, a_i^\dagger - a_i^\dagger a^\dagger(\mathbf{x})\, a(\mathbf{x})\right) && (3)\\
&= \int d^3x\, \left[a^\dagger(\mathbf{x})\left(\delta(\mathbf{x}-\mathbf{x}_i) + a_i^\dagger a(\mathbf{x})\right) - a_i^\dagger a^\dagger(\mathbf{x})\, a(\mathbf{x})\right] && (4)\\
&= \int d^3x\, a^\dagger(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_i) && (5)\\
&= a_i^\dagger && (6)\\
[N, a_i] &= \int d^3x\, \left(a^\dagger(\mathbf{x})\, a(\mathbf{x})\, a_i - a_i a^\dagger(\mathbf{x})\, a(\mathbf{x})\right) && (7)\\
&= \int d^3x\, \left[a^\dagger(\mathbf{x})\, a(\mathbf{x})\, a_i - \left(\delta(\mathbf{x}-\mathbf{x}_i) + a^\dagger(\mathbf{x})\, a_i\right) a(\mathbf{x})\right] && (8)\\
&= -\int d^3x\, a(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_i) && (9)\\
&= -a_i && (10)
\end{aligned}$$

Here we've used the commutation relations

$$\left[a(\mathbf{x}), a(\mathbf{x}')\right] = 0 \tag{11}$$
$$\left[a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')\right] = 0 \tag{12}$$
$$\left[a(\mathbf{x}), a^\dagger(\mathbf{x}')\right] = \delta^3(\mathbf{x}-\mathbf{x}') \tag{13}$$

Now suppose we have an operator X which contains n creation operators $a_i^\dagger$, $i = 1,\ldots,n$ and m annihiliation operators a_j, $j = 1,\ldots,m$:

$$X = a_{i1}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm} \tag{14}$$

Then

$$[N,X] = N a_{i1}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm} - a_{i1}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm} N \tag{15}$$
$$= \left(a_{i1}^\dagger N + a_{i1}^\dagger\right) a_{i2}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm} - a_{i1}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm} N \tag{16}$$
$$= X + a_{i1}^\dagger \left[N, a_{i2}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm}\right] \tag{17}$$

We can see that the commutator in the last line can be worked out recursively until we've processed all the creation operators up to $a_{in}^\dagger$, giving

$$[N,X] = nX + a_{i1}^\dagger \ldots a_{in}^\dagger [N, a_{j1} \ldots a_{jm}] \tag{18}$$

The last commutator gives us

$$[N, a_{j1} \ldots a_{jm}] = N a_{j1} \ldots a_{jm} - a_{j1} \ldots a_{jm} N \tag{19}$$
$$= (a_{j1} N - a_{j1}) a_{j2} \ldots a_{jm} - a_{j1} \ldots a_{jm} N \tag{20}$$
$$= -a_{j1} \ldots a_{jm} + a_{j1} [N, a_{j2} \ldots a_{jm}] \tag{21}$$
$$= -m (a_{j1} \ldots a_{jm}) \tag{22}$$

Therefore

$$a_{i1}^\dagger \ldots a_{in}^\dagger [N, a_{j1} \ldots a_{jm}] = -m\left(a_{i1}^\dagger \ldots a_{in}^\dagger a_{j1} \ldots a_{jm}\right) \tag{23}$$
$$= -mX \tag{24}$$
$$[N,X] = (n-m) X \tag{25}$$

So if $n = m$ (the numbers of creation and annihiliation operators are equal), the operator X commutes with N. In particular, the hamiltonian we met last time satisfies this criterion, so $[N,H] = 0$ and this hamiltonian conserves particle numbers.

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