NUMBER OPERATOR

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 1, Problem 1.3.

The number operator is defined as

$$N\equiv\int d^{3}x\,a^{\dagger}(\mathbf{x})\,a(\mathbf{x}) \tag{1}$$

Applied to a quantum state, it counts the number of particles in that state:

$$Na^{\dagger}(\mathbf{x}_{1})\ldots a^{\dagger}(\mathbf{x}_{n})|0\rangle=na^{\dagger}(\mathbf{x}_{1})\ldots a^{\dagger}(\mathbf{x}_{n})|0\rangle \tag{2}$$

Another property of N is that it commutes with any other operator that contains an equal number of creation and annihilation operators. To see this, look at the individual commutators as follows (where $a_{i}\equiv a(\mathbf{x}_{i})$).

$$\left[N,a_{i}^{\dagger}\right]=\int d^{3}x\,\left(a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\,a_{i}^{\dagger}-a_{i}^{\dagger}a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\right) \tag{3}$$

$$=\int d^{3}x\,\left[a^{\dagger}(\mathbf{x})\left(\delta(\mathbf{x}-\mathbf{x}_{i})+a_{i}^{\dagger}a(\mathbf{x})\right)-a_{i}^{\dagger}a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\right] \tag{4}$$

$$=\int d^{3}x\,a^{\dagger}(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_{i}) \tag{5}$$

$$=a_{i}^{\dagger} \tag{6}$$

$$[N,a_{i}]=\int d^{3}x\,\left(a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\,a_{i}-a_{i}a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\right) \tag{7}$$

$$=\int d^{3}x\,\left[a^{\dagger}(\mathbf{x})\,a(\mathbf{x})\,a_{i}-\left(\delta(\mathbf{x}-\mathbf{x}_{i})+a^{\dagger}(\mathbf{x})\,a_{i}\right)a(\mathbf{x})\right] \tag{8}$$

$$=-\int d^{3}x\,a(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_{i}) \tag{9}$$

$$=-a_{i} \tag{10}$$

Here we've used the commutation relations

$$\left[a\left(\mathbf{x}\right),a\left(\mathbf{x}'\right)\right] = 0 \tag{11}$$

$$\left[a^{\dagger}\left(\mathbf{x}\right),a^{\dagger}\left(\mathbf{x}'\right)\right] = 0 \tag{12}$$

$$\left[a\left(\mathbf{x}\right),a^{\dagger}\left(\mathbf{x}'\right)\right] = \delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{13}$$

Now suppose we have an operator X which contains n creation operators $a_i^{\dagger}$, $i=1,\ldots,n$ and m annihiliation operators a_j, $j=1,\ldots,m$:

$$X = a_{i1}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm} \tag{14}$$

Then

$$[N,X] = Na_{i1}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}-a_{i1}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}N \tag{15}$$

$$= \left(a_{i1}^{\dagger}N+a_{i1}^{\dagger}\right)a_{i2}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}-a_{i1}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}N \tag{16}$$

$$= X+a_{i1}^{\dagger}\left[N,a_{i2}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}\right] \tag{17}$$

We can see that the commutator in the last line can be worked out recursively until we've processed all the creation operators up to $a_{in}^{\dagger}$, giving

$$[N,X] = nX+a_{i1}^{\dagger}\ldots a_{in}^{\dagger}\left[N,a_{j1}\ldots a_{jm}\right] \tag{18}$$

The last commutator gives us

$$\left[N,a_{j1}\ldots a_{jm}\right] = Na_{j1}\ldots a_{jm}-a_{j1}\ldots a_{jm}N \tag{19}$$

$$= \left(a_{j1}N-a_{j1}\right)a_{j2}\ldots a_{jm}-a_{j1}\ldots a_{jm}N \tag{20}$$

$$= -a_{j1}\ldots a_{jm}+a_{j1}\left[N,a_{j2}\ldots a_{jm}\right] \tag{21}$$

$$= -m\left(a_{j1}\ldots a_{jm}\right) \tag{22}$$

Therefore

$$a_{i1}^{\dagger}\ldots a_{in}^{\dagger}\left[N,a_{j1}\ldots a_{jm}\right] = -m\left(a_{i1}^{\dagger}\ldots a_{in}^{\dagger}a_{j1}\ldots a_{jm}\right) \tag{23}$$

$$= -mX \tag{24}$$

$$[N,X] = (n-m)X \tag{25}$$

So if $n=m$ (the numbers of creation and annihiliation operators are equal), the operator X commutes with N. In particular, the hamiltonian we met last time satisfies this criterion, so $[N,H]=0$ and this hamiltonian conserves particle numbers.

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