LORENTZ GROUP

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problem 2.1.

We've seen how to derive the simple Lorentz transformation for the usual introductory case where one inertial frame moves at constant velocity v along the x axis of another inertial frame. We can define a Lorentz transformation more generally as a linear, homogeneous coordinate transformation:

$$\bar{x}^{\mu}=\Lambda_{\nu}^{\mu}x^{\nu} \tag{1}$$

where Λ_{ν}^{μ} is a matrix of constant values (that is, the values don't depend on x^{μ}). In order to preserve the invariant interval in special relativity, Λ_{ν}^{μ} must satisfy the relation

$$g_{\mu\nu}\Lambda_{\rho}^{\mu}\Lambda_{\sigma}^{\nu}=g_{\rho\sigma} \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor, which in Srednicki's book is defined as $g_{00}=-1$ and $g_{ii}=+1$ with off-diagonal elements being zero.

We can see this as follows. The interval between two events x and x' (where these are four-vectors in space-time) is

$$\left(x-x'\right)^{2} = g_{\mu\nu}\left(x-x'\right)^{\mu}\left(x-x'\right)^{\nu} \tag{3}$$
$$= \left(\mathbf{x}-\mathbf{x}'\right)^{2}-c^{2}\left(t-t'\right)^{2} \tag{4}$$

This interval must remain the same after a Lorentz transformation, so we have

$$\left(\bar{x}-\bar{x}'\right)^{2} = g_{\mu\nu}\left(\bar{x}-\bar{x}'\right)^{\mu}\left(\bar{x}-\bar{x}'\right)^{\nu} \tag{5}$$
$$= g_{\mu\nu}\Lambda_{\rho}^{\mu}\Lambda_{\sigma}^{\nu}\left(x-x'\right)^{\rho}\left(x-x'\right)^{\sigma} \tag{6}$$

In order for this to be equal to $\left(x-x'\right)^{2}$ the condition 2 must hold.

The Lorentz transformations defined this way form a mathematical group, which means they satisfy the conditions:

(1) The product of any two Lorentz transformations is another Lorentz transformation.

(2) The product is associative, so that $\Lambda^{\mu}_{\rho}\left(\Lambda^{\nu}_{\sigma}+\Lambda^{\tau}_{\upsilon}\right)=\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}+\Lambda^{\mu}_{\rho}\Lambda^{\tau}_{\upsilon}$.
(3) There is an identity transformation, which is $\Lambda^{\mu}_{\rho}=\delta^{\mu}_{\rho}$.
(4) Every transformation has an inverse.

Srednicki shows how to prove the inverse property, with the result

$$\left(\Lambda^{-1}\right)^{\rho}_{\ \nu}=\Lambda^{\rho}_{\nu} \tag{7}$$

To prove the first property, suppose we multiply 1 on the left by Λ^{ρ}_{μ} to give

$$\Lambda^{\rho}_{\mu}\bar{x}^{\mu}=\Lambda^{\rho}_{\mu}\Lambda^{\mu}_{\nu}x^{\nu} \tag{8}$$

If we consider the product on the RHS as a single transformation and repeat it, then applying 2 we have

$$g_{\sigma\rho}\left(\Lambda^{\sigma}_{\tau}\Lambda^{\tau}_{\pi}\right)\left(\Lambda^{\rho}_{\mu}\Lambda^{\mu}_{\nu}\right)=g_{\sigma\rho}\left(\Lambda^{\sigma}_{\tau}\Lambda^{\rho}_{\mu}\right)\left(\Lambda^{\tau}_{\pi}\Lambda^{\mu}_{\nu}\right) \tag{9}$$
$$=g_{\tau\mu}\Lambda^{\tau}_{\pi}\Lambda^{\mu}_{\nu} \tag{10}$$
$$=g_{\pi\nu} \tag{11}$$

Thus the product $\Lambda^{\sigma}_{\tau}\Lambda^{\tau}_{\pi}$ behaves like a single Lorentz transformation.

For an infinitesimal transformation, we can write it as the identity δ^{μ}_{ν} plus an incremental amount $\delta\omega^{\mu}_{\nu}$. From 2 we must have

$$g_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}=g_{\mu\nu}\left(\delta^{\mu}_{\rho}+\delta\omega^{\mu}_{\rho}\right)\left(\delta^{\nu}_{\sigma}+\delta\omega^{\nu}_{\sigma}\right) \tag{12}$$
$$=g_{\mu\nu}\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}+\delta^{\mu}_{\rho}\delta\omega^{\nu}_{\sigma}+\delta\omega^{\mu}_{\rho}\delta^{\nu}_{\sigma}+\mathcal{O}\left(\delta\omega^{2}\right)\right) \tag{13}$$
$$=g_{\rho\sigma}+\delta\omega_{\rho\sigma}+\delta\omega_{\sigma\rho}+\mathcal{O}\left(\delta\omega^{2}\right) \tag{14}$$

To get the last line, we lowered the indexes on the $\delta\omega$ terms using

$$g_{\mu\nu}\delta\omega^{\nu}_{\sigma}=\delta\omega_{\mu\sigma} \tag{15}$$

and then used the δ^{μ}_{ρ} factor to replace μ by ρ. In order for this to satisfy 2 (to first order in $\delta\omega$), we see that $\delta\omega_{\rho\sigma}$ must be antisymmetric, so that

$$\delta\omega_{\rho\sigma}=-\delta\omega_{\sigma\rho} \tag{16}$$

Since Λ is a 4×4 matrix and, for infinitesimal transformations, the diagonal entries are all 1, this antisymmetry means there are 6 independent elements in the Λ matrix. These 6 degrees of freedom can be interpreted as 3 rotations (about the 3 spatial coordinate axes) and 3 boosts (along the 3 coordinate axes).

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