

LORENTZ GROUP

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problem 2.1.

We've seen how to derive the simple Lorentz transformation for the usual introductory case where one inertial frame moves at constant velocity v along the x axis of another inertial frame. We can define a Lorentz transformation more generally as a linear, homogeneous coordinate transformation:

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu \quad (1)$$

where Λ^μ_ν is a matrix of constant values (that is, the values don't depend on x^μ). In order to preserve the invariant interval in special relativity, Λ^μ_ν must satisfy the relation

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \quad (2)$$

where $g_{\mu\nu}$ is the metric tensor, which in Srednicki's book is defined as $g_{00} = -1$ and $g_{ii} = +1$ with off-diagonal elements being zero.

We can see this as follows. The interval between two events x and x' (where these are four-vectors in space-time) is

$$(x - x')^2 = g_{\mu\nu} (x - x')^\mu (x - x')^\nu \quad (3)$$

$$= (\mathbf{x} - \mathbf{x}')^2 - c^2 (t - t')^2 \quad (4)$$

This interval must remain the same after a Lorentz transformation, so we have

$$(\bar{x} - \bar{x}')^2 = g_{\mu\nu} (\bar{x} - \bar{x}')^\mu (\bar{x} - \bar{x}')^\nu \quad (5)$$

$$= g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma (x - x')^\rho (x - x')^\sigma \quad (6)$$

In order for this to be equal to $(x - x')^2$ the condition 2 must hold.

The Lorentz transformations defined this way form a mathematical group, which means they satisfy the conditions:

- (1) The product of any two Lorentz transformations is another Lorentz transformation.

- (2) The product is associative, so that $\Lambda_\rho^\mu (\Lambda_\sigma^\nu + \Lambda_\nu^\tau) = \Lambda_\rho^\mu \Lambda_\sigma^\nu + \Lambda_\rho^\mu \Lambda_\nu^\tau$.
- (3) There is an identity transformation, which is $\Lambda_\rho^\mu = \delta_\rho^\mu$.
- (4) Every transformation has an inverse.

Srednicki shows how to prove the inverse property, with the result

$$(\Lambda^{-1})_\nu^\rho = \Lambda_\nu^\rho \quad (7)$$

To prove the first property, suppose we multiply 1 on the left by Λ_μ^ρ to give

$$\Lambda_\mu^\rho \bar{x}^\mu = \Lambda_\mu^\rho \Lambda_\nu^\mu x^\nu \quad (8)$$

If we consider the product on the RHS as a single transformation and repeat it, then applying 2 we have

$$g_{\sigma\rho} (\Lambda_\tau^\sigma \Lambda_\pi^\tau) (\Lambda_\mu^\rho \Lambda_\nu^\mu) = g_{\sigma\rho} (\Lambda_\tau^\sigma \Lambda_\mu^\rho) (\Lambda_\pi^\tau \Lambda_\nu^\mu) \quad (9)$$

$$= g_{\tau\mu} \Lambda_\pi^\tau \Lambda_\nu^\mu \quad (10)$$

$$= g_{\pi\nu} \quad (11)$$

Thus the product $\Lambda_\tau^\sigma \Lambda_\pi^\tau$ behaves like a single Lorentz transformation.

For an infinitesimal transformation, we can write it as the identity δ_ν^μ plus an incremental amount $\delta\omega_\nu^\mu$. From 2 we must have

$$g_{\mu\nu} \Lambda_\rho^\mu \Lambda_\sigma^\nu = g_{\mu\nu} (\delta_\rho^\mu + \delta\omega_\rho^\mu) (\delta_\sigma^\nu + \delta\omega_\sigma^\nu) \quad (12)$$

$$= g_{\mu\nu} (\delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\mu \delta\omega_\sigma^\nu + \delta\omega_\rho^\mu \delta_\sigma^\nu + \mathcal{O}(\delta\omega^2)) \quad (13)$$

$$= g_{\rho\sigma} + \delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho} + \mathcal{O}(\delta\omega^2) \quad (14)$$

To get the last line, we lowered the indexes on the $\delta\omega$ terms using

$$g_{\mu\nu} \delta\omega_\sigma^\nu = \delta\omega_{\mu\sigma} \quad (15)$$

and then used the δ_ρ^μ factor to replace μ by ρ . In order for this to satisfy 2 (to first order in $\delta\omega$), we see that $\delta\omega_{\rho\sigma}$ must be antisymmetric, so that

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho} \quad (16)$$

Since Λ is a 4×4 matrix and, for infinitesimal transformations, the diagonal entries are all 1, this antisymmetry means there are 6 independent elements in the Λ matrix. These 6 degrees of freedom can be interpreted as 3 rotations (about the 3 spatial coordinate axes) and 3 boosts (along the 3 coordinate axes).

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