GENERATORS OF THE LORENTZ GROUP

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problem 2.2.

A requirement of any relativistic quantum field theory (or indeed any relativistic theory) is that its equations must be invariant under Lorentz transformations. That is, if we transform from one inertial frame $\mathcal{S}$ to another $\bar{\mathcal{S}}$, the equations describing the system must have the same form in both frames.

A general theorem in quantum mechanics is Wigner's theorem, which states that any symmetry under which a system is invariant can be represented by a unitary or antiunitary transformation. A unitary operator in a vector space is an operator that preserves the norm (length) of all vectors in the space. Unitary operators satisfy a number of other useful conditions, such as $U^{\dagger}=U^{-1}$ and the fact that they preserve inner products, so that $\langle Uu|Uv\rangle=\langle u|v\rangle$ for all u, v in the vector space. An antiunitary operator T is similar, except that for inner products it satisfies $\langle Tu|Tv\rangle=\langle u|v\rangle^{*}=\langle v|u\rangle$.

Thus if a quantum system is symmetric under a Lorentz transformation Λ, there must be an associated unitary operator $U(\Lambda)$. The Lorentz transformations we're considering here are *proper*, in the sense that $\det\Lambda=+1$, and *orthochronous*, meaning that $\Lambda_{0}^{0}\geq+1$. Because the product of two Lorentz transformations is another Lorentz transformation, the unitary operator corresponding to the product must be the result of applying two successive Lorentz transformations, so that

$$U\left(\Lambda'\Lambda\right)=U\left(\Lambda'\right)U\left(\Lambda\right) \tag{1}$$

For an infinitesimal transformation $\Lambda'=I+\delta\omega$, Srednicki states that we can write

$$U\left(I+\delta\omega\right)=I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu} \tag{2}$$

Srednicki actually writes $\Lambda'=1+\delta\omega'$ but he drops the prime on ω in eqn 2.13, and I think it's better to use I instead of 1 since Λ' is actually a matrix.

where $M^{\mu\nu}=-M^{\nu\mu}$. The reason we can choose $M^{\mu\nu}$ to be antisymmetric is that it appears only in a total sum with $\delta\omega_{\mu\nu}$ which is antisymmetric. Any 2nd-rank tensor can be written as a sum of symmetric and antisymmetric parts, according to

$$M^{\mu\nu}=\frac{1}{2}\left(M^{\mu\nu}+M^{\nu\mu}\right)+\frac{1}{2}\left(M^{\mu\nu}-M^{\nu\mu}\right) \tag{3}$$

and the sum of $\delta\omega_{\mu\nu}$ multiplied by the symmetric part is zero, so we might as well choose $M^{\mu\nu}$ to be antisymmetric.

The $M^{\mu\nu}$ are also assumed to be hermitian, and are known as the *generators of the Lorentz group*.

In any case, if we accept this definition, we can find how the generators transform. We begin with

$$U(\Lambda)^{-1}U\left(\Lambda'\right)U(\Lambda)=U\left(\Lambda^{-1}\Lambda'\Lambda\right) \tag{4}$$

which follows from successive application of 1. We now take $\Lambda'=I+\delta\omega$ to be an infinitesimal transformation (note that Λ is an ordinary Lorentz transformation, that is, not necessarily infinitesimal). Expanding the LHS of 4 we get

$$\begin{aligned}
U(\Lambda)^{-1}U\left(\Lambda'\right)U(\Lambda) &= U(\Lambda)^{-1}U(I+\delta\omega)U(\Lambda) && (5)\\
&= U(\Lambda)^{-1}\left(I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) && (6)\\
&= U(\Lambda)^{-1}U(\Lambda)+\frac{i}{2\hbar}U(\Lambda)^{-1}\delta\omega_{\mu\nu}M^{\mu\nu}U(\Lambda) && (7)\\
&= I+\frac{i}{2\hbar}U(\Lambda)^{-1}\delta\omega_{\mu\nu}M^{\mu\nu}U(\Lambda) && (8)
\end{aligned}$$

The RHS of 4 gives us

$$\begin{aligned}
U\left(\Lambda^{-1}\Lambda'\Lambda\right) &= U\left(\Lambda^{-1}(I+\delta\omega)\Lambda\right) && (9)\\
&= U\left(I+\Lambda^{-1}\delta\omega\Lambda\right) && (10)
\end{aligned}$$

Because of the $\delta\omega$ factor, the overall transformation $\Lambda^{-1}\Lambda'\Lambda$ is an infinitesimal transformation, so we can apply 2 to get

$$U\left(I+\Lambda^{-1}\delta\omega\Lambda\right)=I+\frac{i}{2\hbar}\left(\Lambda^{-1}\delta\omega\Lambda\right)_{\rho\sigma}M^{\rho\sigma} \tag{11}$$

We therefore need to work out the matrix elements of $\Lambda^{-1}\delta\omega\Lambda$. To get the indices in the right places, we note first of all that in order to match the RHS with the LHS of 4, we want the indexes of $\delta\omega$ to be in the same places on both sides of the equation. In 8, we have $\delta\omega_{\mu\nu}$ with both indexes lowered, so we'd like $\delta\omega_{\mu\nu}$ to appear on the RHS as well.

The object $\Lambda^{-1}\delta\omega\Lambda$ is the product of 3 matrices. The left product $\Lambda^{-1}\delta\omega$ multiplies the columns (second index) of Λ^{-1} into the rows (first index) of

$\delta\omega$, and the row (first) index of Λ^{-1} will become the index ρ of the overall product, so we can write the elements of $\Lambda^{-1}\delta\omega$ as $\left(\Lambda^{-1}\right)_{\rho}^{\ \mu}\delta\omega_{\mu\nu}$. We now multiply the columns (index ν) of this product into the rows (first index) of Λ. Since ν is a lower index of $\left(\Lambda^{-1}\right)_{\rho}^{\ \mu}\delta\omega_{\mu\nu}$ it must be an upper index of Λ in order for the summation convention to apply, so the final result is

$$\left(\Lambda^{-1}\delta\omega\Lambda\right)_{\rho\sigma}=\left(\Lambda^{-1}\right)_{\rho}^{\ \mu}\delta\omega_{\mu\nu}\Lambda_{\ \sigma}^{\nu} \tag{12}$$

We can now use the earlier result to get rid of the Λ^{-1} term:

$$\left(\Lambda^{-1}\right)_{\ \mu}^{\rho}=\Lambda_{\mu}^{\ \rho} \tag{13}$$

By lowering the ρ and raising the μ on both sides, we get

$$\left(\Lambda^{-1}\right)_{\rho}^{\ \mu}=\Lambda_{\ \rho}^{\mu} \tag{14}$$

so

$$\left(\Lambda^{-1}\delta\omega\Lambda\right)_{\rho\sigma}=\Lambda_{\ \rho}^{\mu}\delta\omega_{\mu\nu}\Lambda_{\ \sigma}^{\nu} \tag{15}$$

$$=\delta\omega_{\mu\nu}\Lambda_{\ \sigma}^{\nu}\Lambda_{\ \rho}^{\mu} \tag{16}$$

Plugging this back into 11 and equating this with 8 we get

$$I+\frac{i}{2\hbar}U\left(\Lambda\right)^{-1}\delta\omega_{\mu\nu}M^{\mu\nu}U\left(\Lambda\right)=I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}\Lambda_{\ \sigma}^{\nu}\Lambda_{\ \rho}^{\mu}M^{\rho\sigma} \tag{17}$$

Cancelling terms gives

$$U\left(\Lambda\right)^{-1}\delta\omega_{\mu\nu}M^{\mu\nu}U\left(\Lambda\right)=\delta\omega_{\mu\nu}\Lambda_{\ \sigma}^{\nu}\Lambda_{\ \rho}^{\mu}M^{\rho\sigma} \tag{18}$$

Since the infinitesimal transformation components $\delta\omega_{\mu\nu}$ are arbitrary, their coefficients on each side must be equal, so we get the final result:

$$U\left(\Lambda\right)^{-1}M^{\mu\nu}U\left(\Lambda\right)=\Lambda_{\ \sigma}^{\nu}\Lambda_{\ \rho}^{\mu}M^{\rho\sigma} \tag{19}$$

Thus each index on the generator $M^{\rho\sigma}$ undergoes its own Lorentz transformation.

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