

GENERATORS OF THE LORENTZ GROUP - COMMUTATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 19 Mar 2018.

References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problems 2.3 - 2.4.

The generators of the Lorentz group were defined from the unitary operator for the Lorentz group:

$$U(I + \delta\omega) = I + \frac{i}{2\hbar} \delta\omega_{\mu\nu} M^{\mu\nu} \quad (1)$$

where $M^{\mu\nu}$ are the generators. That satisfy the transformation

$$U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^\nu_\sigma \Lambda^\mu_\rho M^{\rho\sigma} \quad (2)$$

By considering an infinitesimal transformation

$$\Lambda = 1 + \delta\omega \quad (3)$$

we can work out the commutators. As usual, we'll work to first order in $\delta\omega$. Substituting 1 into the LHS of 2 we have

$$\begin{aligned} U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) &= \left(1 - \frac{i}{2\hbar} \delta\omega_{\alpha\beta} M^{\alpha\beta}\right) M^{\mu\nu} \left(1 - \frac{i}{2\hbar} \delta\omega_{\gamma\delta} M^{\gamma\delta}\right) \quad (4) \\ &= M^{\mu\nu} + \frac{i}{2\hbar} \left(-\delta\omega_{\alpha\beta} M^{\alpha\beta} M^{\mu\nu} + M^{\mu\nu} \delta\omega_{\gamma\delta} M^{\gamma\delta}\right) \quad (5) \end{aligned}$$

The summation indexes on the terms in the parentheses are dummy indexes, so we can combine the two sums to get

$$U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = M^{\mu\nu} + \frac{i}{2\hbar} \delta\omega_{\rho\sigma} (M^{\mu\nu} M^{\rho\sigma} - M^{\rho\sigma} M^{\mu\nu}) \quad (6)$$

$$= M^{\mu\nu} + \frac{i}{2\hbar} \delta\omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] \quad (7)$$

where in the first line we made the replacements $\alpha \rightarrow \rho$, $\beta \rightarrow \sigma$, $\gamma \rightarrow \rho$ and $\delta \rightarrow \sigma$. The bracket in the last line indicates the commutator.

Now consider the RHS of 2. We need to be careful with the indexes here, so we'll write

$$\Lambda_\sigma^\nu = \delta_\sigma^\nu + \delta\omega_\sigma^\nu \quad (8)$$

$$\Lambda_\rho^\mu = \delta_\rho^\mu + \delta\omega_\rho^\mu \quad (9)$$

Then the RHS of 2 becomes

$$\Lambda_\sigma^\nu \Lambda_\rho^\mu M^{\rho\sigma} = (\delta_\sigma^\nu + \delta\omega_\sigma^\nu) (\delta_\rho^\mu + \delta\omega_\rho^\mu) M^{\rho\sigma} \quad (10)$$

$$= \delta_\sigma^\nu \delta_\rho^\mu M^{\rho\sigma} + (\delta_\sigma^\nu \delta\omega_\rho^\mu + \delta\omega_\sigma^\nu \delta_\rho^\mu) M^{\rho\sigma} \quad (11)$$

$$= M^{\mu\nu} + (\delta\omega_\rho^\mu M^{\rho\nu} + \delta\omega_\sigma^\nu M^{\mu\sigma}) \quad (12)$$

$$= M^{\mu\nu} + (g^{\mu\sigma} \delta\omega_{\sigma\rho} M^{\rho\nu} + g^{\nu\rho} \delta\omega_{\rho\sigma} M^{\mu\sigma}) \quad (13)$$

To compare this with 7, we would like the $\delta\omega$ terms to have the same indexes on both sides. Since $\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$, we can rewrite 13 as

$$\Lambda_\sigma^\nu \Lambda_\rho^\mu M^{\rho\sigma} = M^{\mu\nu} + \delta\omega_{\rho\sigma} (g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\rho\nu}) \quad (14)$$

Setting this equal to 7 we can cancel the $M^{\mu\nu}$ term to get

$$\frac{i}{2\hbar} \delta\omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] = \delta\omega_{\rho\sigma} (g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\rho\nu}) \quad (15)$$

Now since both $\delta\omega_{\rho\sigma}$ and $M^{\rho\sigma}$ are antisymmetric, if we swap $\rho \leftrightarrow \sigma$ in 15, the result is unchanged. That is, we can add 15 to its swapped version to get

$$\frac{i}{2\hbar} \delta\omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] + \frac{i}{2\hbar} \delta\omega_{\sigma\rho} [M^{\mu\nu}, M^{\sigma\rho}] = \frac{i}{\hbar} \delta\omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] \quad (16)$$

However, if we also swap $\rho \leftrightarrow \sigma$ in 14 and add to the original we get

$$\frac{i}{\hbar} \delta\omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] = \delta\omega_{\rho\sigma} (g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\rho\nu}) + \quad (17)$$

$$\delta\omega_{\sigma\rho} (g^{\nu\sigma} M^{\mu\rho} - g^{\mu\rho} M^{\sigma\nu}) \quad (18)$$

$$= \delta\omega_{\rho\sigma} ((g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\rho\nu}) - (g^{\nu\sigma} M^{\mu\rho} - g^{\mu\rho} M^{\sigma\nu})) \quad (19)$$

where we used $\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$ to get the last line. If we multiply through by $-i\hbar$ and use the fact that $\delta\omega_{\rho\sigma}$ is arbitrary so we can equate its coefficients, we get (using $M^{\sigma\nu} = -M^{\nu\sigma}$ and $M^{\rho\nu} = -M^{\nu\rho}$):

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar ((g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\rho\nu}) - (g^{\nu\sigma} M^{\mu\rho} - g^{\mu\rho} M^{\sigma\nu})) \quad (20)$$

$$= i\hbar ((g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma}) - (g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho})) \quad (21)$$

which is eqn 2.16 in Srednicki.

We now propose that the angular momentum operators are

$$J_i = \frac{1}{2} \varepsilon_{ijk} M^{jk} \quad (22)$$

Note that this implies a sum over j and k from 1 to 3.

We also define the boost operators as

$$K_i = M^{i0} \quad (23)$$

We can now use 21 to work out the commutators involving J_i and K_i . Consider

$$[J_1, J_2] = \frac{1}{4} [M^{23} - M^{32}, M^{31} - M^{13}] \quad (24)$$

$$= \frac{1}{4} ([M^{23}, M^{31}] - [M^{23}, M^{13}] - [M^{32}, M^{31}] + [M^{32}, M^{13}]) \quad (25)$$

The commutators are worked out by using 21 with appropriate values for the various indexes. Remember that $g^{\mu\nu}$ is diagonal, so it's equal to 0 unless $\mu = \nu$. We have

$$[M^{23}, M^{31}] = -i\hbar g^{33} M^{21} = i\hbar M^{12} \quad (26)$$

$$[M^{23}, M^{13}] = +i\hbar g^{33} M^{21} = -i\hbar M^{12} \quad (27)$$

$$[M^{32}, M^{31}] = +i\hbar g^{33} M^{21} = -i\hbar M^{12} \quad (28)$$

$$[M^{32}, M^{13}] = -i\hbar g^{33} M^{21} = i\hbar M^{12} \quad (29)$$

Putting them all into 25 we end up with

$$[J_1, J_2] = i\hbar M^{12} = \frac{i\hbar}{2} (M^{12} - M^{21}) = i\hbar \frac{1}{2} \varepsilon_{3jk} M^{jk} = i\hbar J_3 \quad (30)$$

The other commutators can be worked out the same way, and the general formula is the standard angular momentum commutator

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \quad (31)$$

We now have

$$[J_1, K_2] = \frac{1}{2} [M^{23} - M^{32}, M^{20}] \quad (32)$$

$$= \frac{i\hbar}{2} (M^{30} - (-M^{30})) \quad (33)$$

$$= i\hbar M^{30} = i\hbar K_3 \quad (34)$$

The general formula is

There may be some clever way of proving the result in general, but I can't see it immediately.

$$[J_i, K_j] = i\hbar \varepsilon_{ijk} K_k \quad (35)$$

Finally, we have

$$[K_1, K_2] = i\hbar [M^{10}, M^{20}] \quad (36)$$

$$= i\hbar g^{00} M^{12} \quad (37)$$

$$= -i\hbar M^{12} \quad (38)$$

$$= -\frac{i\hbar}{2} (M^{12} - M^{21}) \quad (39)$$

$$= -i\hbar J_3 \quad (40)$$

The general formula is

$$[K_i, K_j] = -i\hbar \varepsilon_{ijk} J_k \quad (41)$$

PINGBACKS

Pingback: [Generators of the Lorentz group - Momentum Commutators](#)

Pingback: [Generators of the Lorentz group - alternative derivation of the Commutators](#)