GENERATORS OF THE LORENTZ GROUP - COMMUTATORS

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 19 Mar 2018.
References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problems 2.3 - 2.4.

The generators of the Lorentz group were defined from the unitary operator for the Lorentz group:

$$U(I+\delta\omega)=I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu} \tag{1}$$

where $M^{\mu\nu}$ are the generators. That satisfy the transformation

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda)=\Lambda_{\sigma}^{\nu}\Lambda_{\rho}^{\mu}M^{\rho\sigma} \tag{2}$$

By considering an infinitesimal transformation

$$\Lambda=1+\delta\omega \tag{3}$$

we can work out the commutators. As usual, we'll work to first order in $\delta\omega$. Substituting 1 into the LHS of 2 we have

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda)=\left(1-\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(1-\frac{i}{2\hbar}\delta\omega_{\gamma\delta}M^{\gamma\delta}\right) \tag{4}$$

$$=M^{\mu\nu}+\frac{i}{2\hbar}\left(-\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu}+M^{\mu\nu}\delta\omega_{\gamma\delta}M^{\gamma\delta}\right) \tag{5}$$

The summation indexes on the terms in the parentheses are dummy indexes, so we can combine the two sums to get

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda)=M^{\mu\nu}+\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left(M^{\mu\nu}M^{\rho\sigma}-M^{\rho\sigma}M^{\mu\nu}\right) \tag{6}$$

$$=M^{\mu\nu}+\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu},M^{\rho\sigma}\right] \tag{7}$$

where in the first line we made the replacements $\alpha\to\rho$, $\beta\to\sigma$, $\gamma\to\rho$ and $\delta\to\sigma$. The bracket in the last line indicates the commutator.

Now consider the RHS of 2. We need to be careful with the indexes here, so we'll write

$$\Lambda_{\sigma}^{\nu} = \delta_{\sigma}^{\nu} + \delta\omega_{\sigma}^{\nu} \tag{8}$$
$$\Lambda_{\rho}^{\mu} = \delta_{\rho}^{\mu} + \delta\omega_{\rho}^{\mu} \tag{9}$$

Then the RHS of 2 becomes

$$\begin{aligned}
\Lambda_{\sigma}^{\nu}\Lambda_{\rho}^{\mu}M^{\rho\sigma} &= \left(\delta_{\sigma}^{\nu} + \delta\omega_{\sigma}^{\nu}\right)\left(\delta_{\rho}^{\mu} + \delta\omega_{\rho}^{\mu}\right)M^{\rho\sigma} & (10)\\
&= \delta_{\sigma}^{\nu}\delta_{\rho}^{\mu}M^{\rho\sigma} + \left(\delta_{\sigma}^{\nu}\delta\omega_{\rho}^{\mu} + \delta\omega_{\sigma}^{\nu}\delta_{\rho}^{\mu}\right)M^{\rho\sigma} & (11)\\
&= M^{\mu\nu} + \left(\delta\omega_{\rho}^{\mu}M^{\rho\nu} + \delta\omega_{\sigma}^{\nu}M^{\mu\sigma}\right) & (12)\\
&= M^{\mu\nu} + \left(g^{\mu\sigma}\delta\omega_{\sigma\rho}M^{\rho\nu} + g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma}\right) & (13)
\end{aligned}$$

To compare this with 7, we would like the $\delta\omega$ terms to have the same indexes on both sides. Since $\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$, we can rewrite 13 as

$$\Lambda_{\sigma}^{\nu}\Lambda_{\rho}^{\mu}M^{\rho\sigma} = M^{\mu\nu} + \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\rho\nu}\right) \tag{14}$$

Setting this equal to 7 we can cancel the $M^{\mu\nu}$ term to get

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right] = \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\rho\nu}\right) \tag{15}$$

Now since both $\delta\omega_{\rho\sigma}$ and $M^{\rho\sigma}$ are antisymmetric, if we swap $\rho \leftrightarrow \sigma$ in 15, the result is unchanged. That is, we can add 15 to its swapped version to get

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right] + \frac{i}{2\hbar}\delta\omega_{\sigma\rho}\left[M^{\mu\nu}, M^{\sigma\rho}\right] = \frac{i}{\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right] \tag{16}$$

However, if we also swap $\rho \leftrightarrow \sigma$ in 14 and add to the original we get

$$\begin{aligned}
\frac{i}{\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right] &= \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\rho\nu}\right) + & (17)\\
&\quad \delta\omega_{\sigma\rho}\left(g^{\nu\sigma}M^{\mu\rho} - g^{\mu\rho}M^{\sigma\nu}\right) & (18)\\
&= \delta\omega_{\rho\sigma}\left(\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\rho\nu}\right) - \left(g^{\nu\sigma}M^{\mu\rho} - g^{\mu\rho}M^{\sigma\nu}\right)\right) & (19)
\end{aligned}$$

where we used $\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$ to get the last line. If we multiply through by $-i\hbar$ and use the fact that $\delta\omega_{\rho\sigma}$ is arbitrary so we can equate its coefficients, we get (using $M^{\sigma\nu} = -M^{\nu\sigma}$ and $M^{\rho\nu} = -M^{\nu\rho}$):

$$\begin{aligned}
\left[M^{\mu\nu}, M^{\rho\sigma}\right] &= -i\hbar\left(\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\rho\nu}\right) - \left(g^{\nu\sigma}M^{\mu\rho} - g^{\mu\rho}M^{\sigma\nu}\right)\right) & (20)\\
&= i\hbar\left(\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \left(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}\right)\right) & (21)
\end{aligned}$$

which is eqn 2.16 in Srednicki.

We now propose that the angular momentum operators are

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \tag{22}$$

Note that this implies a sum over j and k from 1 to 3.

We also define the boost operators as

$$K_i = M^{i0} \tag{23}$$

We can now use 21 to work out the commutators involving J_i and K_i. Consider

$$[J_1, J_2] = \frac{1}{4}\left[M^{23} - M^{32}, M^{31} - M^{13}\right] \tag{24}$$

$$= \frac{1}{4}\left(\left[M^{23}, M^{31}\right] - \left[M^{23}, M^{13}\right] - \left[M^{32}, M^{31}\right] + \left[M^{32}, M^{13}\right]\right) \tag{25}$$

The commutators are worked out by using 21 with appropriate values for the various indexes. Remember that $g^{\mu\nu}$ is diagonal, so it's equal to 0 unless $\mu = \nu$. We have

$$\left[M^{23}, M^{31}\right] = -i\hbar g^{33}M^{21} = i\hbar M^{12} \tag{26}$$

$$\left[M^{23}, M^{13}\right] = +i\hbar g^{33}M^{21} = -i\hbar M^{12} \tag{27}$$

$$\left[M^{32}, M^{31}\right] = +i\hbar g^{33}M^{21} = -i\hbar M^{12} \tag{28}$$

$$\left[M^{32}, M^{13}\right] = -i\hbar g^{33}M^{21} = i\hbar M^{12} \tag{29}$$

Putting them all into 25 we end up with

$$[J_1, J_2] = i\hbar M^{12} = \frac{i\hbar}{2}\left(M^{12} - M^{21}\right) = i\hbar\frac{1}{2}\epsilon_{3jk}M^{jk} = i\hbar J_3 \tag{30}$$

The other commutators can be worked out the same way, and the general formula is the standard angular momentum commutator

There may be some clever way of proving the result in general, but I can't see it immediately.

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{31}$$

We now have

$$[J_1, K_2] = \frac{1}{2}\left[M^{23} - M^{32}, M^{20}\right] \tag{32}$$

$$= \frac{i\hbar}{2}\left(M^{30} - \left(-M^{30}\right)\right) \tag{33}$$

$$= i\hbar M^{30} = i\hbar K_3 \tag{34}$$

The general formula is

$$[J_i, K_j] = i\hbar\epsilon_{ijk}K_k \tag{35}$$

Finally, we have

$$\begin{aligned}
[K_1, K_2] &= i\hbar\left[M^{10}, M^{20}\right] &(36)\\
&= i\hbar g^{00}M^{12} &(37)\\
&= -i\hbar M^{12} &(38)\\
&= -\frac{i\hbar}{2}\left(M^{12} - M^{21}\right) &(39)\\
&= -i\hbar J_3 &(40)
\end{aligned}$$

The general formula is

$$[K_i, K_j] = -i\hbar\epsilon_{ijk}J_k \tag{41}$$

PINGBACKS

Pingback: Generators of the Lorentz group - Momentum Commutators
Pingback: Generators of the Lorentz group - alternative derivation of the Commutators