GENERATORS OF THE LORENTZ GROUP - MOMENTUM COMMUTATORS

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problems 2.5 - 2.6.

The four-momentum P^{μ} transforms under a Lorentz transformation according to Srednicki's equation 2.15:

$$U^{-1}(\Lambda)\,P^{\mu}U(\Lambda)=\Lambda_{\sigma}^{\mu}P^{\sigma} \tag{1}$$

Under an infinitesimal transformation, the unitary operators have the form

$$U(I+\delta\omega)=I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu} \tag{2}$$

By using the same technique that we used to find the commutators of $M^{\mu\nu}$ we can find the commutators of P^{μ}. The LHS of 1 is, to first order in $\delta\omega$:

$$U^{-1}(\Lambda)\,P^{\mu}U(\Lambda)=\left(I-\frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)P^{\mu}\left(I+\frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{3}$$

$$=P^{\mu}+\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[P^{\mu},M^{\rho\sigma}\right] \tag{4}$$

The RHS of 1 is

$$\Lambda_{\nu}^{\mu}P^{\nu}=\left(\delta_{\nu}^{\mu}+\delta\omega_{\nu}^{\mu}\right)P^{\nu} \tag{5}$$

$$=P^{\mu}+g^{\mu\rho}\delta\omega_{\rho\sigma}P^{\sigma} \tag{6}$$

Comparing with 4 we can cancel the P^{μ} term on both sides. The remaining term on each side has the form of $\delta\omega_{\rho\sigma}$ multiplied by a coefficient. The term $\delta\omega_{\rho\sigma}\left[P^{\mu},M^{\rho\sigma}\right]$ in 4 is symmetric under the interchange $\rho\leftrightarrow\sigma$ (since both $\delta\omega_{\rho\sigma}$ and $M^{\rho\sigma}$ are antisymmetric), so swapping $\rho\leftrightarrow\sigma$ leaves this term unchanged. Swapping $\rho\leftrightarrow\sigma$ in 6 does give us something different however. If we swap $\rho\leftrightarrow\sigma$ in both equations and then add the results we get

$$\frac{i}{\hbar}\delta\omega_{\rho\sigma}\left[P^{\mu},M^{\rho\sigma}\right]=g^{\mu\rho}\delta\omega_{\rho\sigma}P^{\sigma}+g^{\mu\sigma}\delta\omega_{\sigma\rho}P^{\rho} \tag{7}$$

$$=g^{\mu\rho}\delta\omega_{\rho\sigma}P^{\sigma}-g^{\mu\sigma}\delta\omega_{\rho\sigma}P^{\rho} \tag{8}$$

$$=\delta\omega_{\rho\sigma}\left(g^{\mu\rho}P^{\sigma}-g^{\mu\sigma}P^{\rho}\right) \tag{9}$$

Since $\delta\omega_{\rho\sigma}$ is arbitrary, its coefficients must be equal on each side of the equation. Multiplying through by $-i\hbar$ we get Srednicki's equation 2.18:

$$\left[P^{\mu},M^{\rho\sigma}\right]=i\hbar\left(g^{\mu\sigma}P^{\rho}-g^{\mu\rho}P^{\sigma}\right) \tag{10}$$

Using this formula, we can work out the commutators of the angular momentum and boost operators with P^{μ}. We recall that

$$J_i=\frac{1}{2}\varepsilon_{ijk}M^{jk} \tag{11}$$

$$K_i=M^{i0} \tag{12}$$

The energy H is cP^0 so using 10 we have, for example

$$\left[J_1,H\right]=\frac{c}{2}\left[M^{23}-M^{32},P^0\right] \tag{13}$$

$$=c\left[M^{23},P^0\right] \tag{14}$$

$$=-c\left[P^0,M^{23}\right] \tag{15}$$

$$=0 \tag{16}$$

where the last line follows because we set $\mu=0$, $\rho=2$ and $\sigma=3$ in 10, which causes all the g terms to be zero, since $g^{\mu\nu}$ is diagonal. The same result follows for $[J_2,H]=[J_3,H]=0$.

Now consider

$$\left[J_1,P_2\right]=\left[M^{23},P^2\right] \tag{17}$$

$$=-i\hbar\left(g^{23}P^2-g^{22}P^3\right) \tag{18}$$

$$=i\hbar P^3 \tag{19}$$

We can cycle the indexes to get the other commutation relations with the general result (for spatial indexes 1,2,3, $P^i=P_i$):

$$\left[J_i,P_j\right]=i\hbar\varepsilon_{ijk}P_k \tag{20}$$

For the boosts we have

$$[K_1, H] = c\left[M^{10}, P^0\right] \tag{21}$$
$$= -ic\hbar\left(g^{00}P^1 - g^{01}P^0\right) \tag{22}$$
$$= ic\hbar P^1 \tag{23}$$

since $g^{00} = -1$. Cyclic permutation gives the general result

$$[K_i, H] = ic\hbar P_i \tag{24}$$

Finally we consider

$$[K_1, P_2] = \left[M^{10}, P^2\right] \tag{25}$$
$$= -i\hbar\left(g^{20}P^1 - g^{21}P^0\right) \tag{26}$$
$$= 0 \tag{27}$$

and

$$[K_1, P_1] = \left[M^{10}, P^1\right] \tag{28}$$
$$= -i\hbar\left(g^{10}P^1 - g^{11}P^0\right) \tag{29}$$
$$= i\hbar P^0 \tag{30}$$
$$= i\hbar\frac{H}{c} \tag{31}$$

The general result is

$$[K_i, P_j] = \frac{i\hbar}{c}\delta_{ij}H \tag{32}$$