GENERATORS OF THE LORENTZ GROUP - ALTERNATIVE DERIVATION OF THE COMMUTATORS

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References: Mark Srednicki, *Quantum Field Theory*, (Cambridge University Press, 2007) - Chapter 2, Problem 2.8.

In the Heisenberg picture, the time dependence of a quantum system resides in the operators, rather than in the wave functions or states. In nonrelativistic theory, the time evolution of a state is given by

$$\phi(\mathbf{x},t) = e^{iHt/\hbar}\phi(\mathbf{x},0)\,e^{-iHt/\hbar} \tag{1}$$

where H is the hamiltonian. The relativistic generalization is

$$\phi(x) = e^{-iPx/\hbar}\phi(0)\,e^{iPx/\hbar} \tag{2}$$

where P and x are the momentum and spacetime four-vectors. By defining the spacetime translation operator as

$$T(a) \equiv \exp\left(-iP^{\mu}a_{\mu}/\hbar\right) \tag{3}$$

where a_{μ} is a spacetime four-vector we can write 2 as

$$\phi(a) = T(a)\,\phi(0)\,T^{-1}(a) \tag{4}$$

or, if we start at location $x-a$, the translation $T(a)$ moves us to location x. We can write the inverse of this transformation as

$$\phi(x-a) = T^{-1}(a)\,\phi(x)\,T(a) \tag{5}$$

Srednicki then draws an analogy between this general spacetime transformation and the Lorentz transformation to write his equation 2.26, where we have, for a Lorentz transformation Λ

$$U^{-1}(\Lambda)\,\phi(x)\,U(\Lambda) = \phi\left(\Lambda^{-1}x\right) \tag{6}$$

Another way of writing this to get a forward transformation is

$$\phi(x) = U(\Lambda)\,\phi\left(\Lambda^{-1}x\right)U^{-1}(\Lambda) \tag{7}$$

We've seen that for an infinitesimal transformation we can write $U(\Lambda)$ as

$$U(I+\delta\omega)=I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu} \tag{8}$$

where $M^{\mu\nu}=-M^{\nu\mu}$ are the generators of the Lorentz group. We've also seen that the commutators are given by

$$[M^{\mu\nu},M^{\rho\sigma}]=i\hbar\left(\left(g^{\mu\rho}M^{\nu\sigma}-g^{\nu\rho}M^{\mu\sigma}\right)-\left(g^{\mu\sigma}M^{\nu\rho}-g^{\nu\sigma}M^{\mu\rho}\right)\right) \tag{9}$$

Another way of deriving this begins with 6 for an infinitesimal transformation. We have

$$\left(I-\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\phi(x)\left(I+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)=\phi\left(\left(I-\delta\omega_{\mu\nu}\right)x^{\nu}\right) \tag{10}$$

The LHS can be expanded in the same way we used earlier to get, to first order in $\delta\omega_{\mu\nu}$

$$LHS=\phi(x)+\frac{i}{2\hbar}\delta\omega_{\mu\nu}\left(\phi(x)M^{\mu\nu}-M^{\mu\nu}\phi(x)\right) \tag{11}$$

The RHS of 10 can be expanded in a Taylor series to the same order:

$$\phi\left(\left(I-\delta\omega_{\mu\nu}\right)x\right)=\phi(x)-\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x) \tag{12}$$

We can cancel the $\phi(x)$ from both sides to get

$$\frac{i}{2\hbar}\delta\omega_{\mu\nu}\left(\phi(x)M^{\mu\nu}-M^{\mu\nu}\phi(x)\right)=-\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x) \tag{13}$$

. Because both $\delta\omega_{\mu\nu}$ and $M^{\mu\nu}$ are antisymmetric, we can swap $\mu\leftrightarrow\nu$ on the LHS of this equation, leaving it unchanged. However, the RHS does change under this swap, so if we add the original equation to its swapped counterpart, we get

$$\frac{i}{\hbar}\delta\omega_{\mu\nu}\left(\phi(x)M^{\mu\nu}-M^{\mu\nu}\phi(x)\right)=-\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)-\delta\omega_{\nu\mu}x^{\mu}\partial^{\nu}\phi(x) \tag{14}$$

$$=-\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)+\delta\omega_{\mu\nu}x^{\mu}\partial^{\nu}\phi(x) \tag{15}$$

$$=\delta\omega_{\mu\nu}\left(x^{\mu}\partial^{\nu}-x^{\nu}\partial^{\mu}\right)\phi(x) \tag{16}$$

Multiplying through by $\frac{\hbar}{i}$ and equating coefficients of $\delta\omega_{\mu\nu}$ we get

$$\phi(x)M^{\mu\nu}-M^{\mu\nu}\phi(x)=\frac{\hbar}{i}\left(x^{\mu}\partial^{\nu}-x^{\nu}\partial^{\mu}\right)\phi(x) \tag{17}$$

or

$$[\phi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x) = \mathcal{L}^{\mu\nu}\phi(x) \tag{18}$$

where

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{19}$$

We can now work out the following.

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = (\mathcal{L}^{\mu\nu}\phi(x))M^{\rho\sigma} - M^{\rho\sigma}\mathcal{L}^{\mu\nu}\phi(x) \tag{20}$$

$$= \mathcal{L}^{\mu\nu}[\phi(x), M^{\rho\sigma}] \tag{21}$$

$$= \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x) \tag{22}$$

We're justified in taking $\mathcal{L}^{\mu\nu}$ outside the commutator in the second line, since $\mathcal{L}^{\mu\nu}$ operates only on functions of x, and $M^{\rho\sigma}$ does not depend on x.

Srednicki then asks us to prove the Jacobi identity for the commutators of three operators, which is

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0 \tag{23}$$

This can be proved by brute force by just writing out all the commutators in full and then finding that the terms cancel in pairs. I won't bother with this as it gets quite tedious. Just note that, for example

$$[[A,B],C] = [A,B]C - C[A,B] \tag{24}$$

$$= ABC - BAC - CAB + CBA \tag{25}$$

and so on for the other two.

We can now use 22 and 23 to derive the following.

$$[\phi, [M^{\mu\nu}, M^{\rho\sigma}]] = -[[M^{\mu\nu}, M^{\rho\sigma}], \phi] \tag{26}$$

$$= [[M^{\rho\sigma}, \phi], M^{\mu\nu}] + [[\phi, M^{\mu\nu}], M^{\rho\sigma}] \tag{27}$$

$$= -\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\phi(x) + \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x) \tag{28}$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\phi(x) \tag{29}$$

To simplify this, we need to work out the $\mathcal{L}$ operators as they act on $\phi(x)$, using its definition 19. To do this, we first note that, since the x^ν are independent variables

$$\partial^\mu x^\nu = g^{\mu\nu} \tag{30}$$

We can do the tedious derivatives, using the product rule where required. For the first term in 29 we have

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x) = -\hbar^2\left[x^\mu\left(g^{\nu\rho}\partial^\sigma\phi + x^\rho\partial^{\nu\sigma}\phi - g^{\nu\sigma}\partial^\rho\phi - x^\sigma\partial^{\nu\rho}\phi\right)\right] \quad (31)$$
$$+\hbar^2\left[x^\nu\left(g^{\mu\rho}\partial^\sigma\phi + x^\rho\partial^{\mu\sigma}\phi - g^{\mu\sigma}\partial^\rho\phi - x^\sigma\partial^{\mu\rho}\phi\right)\right] \quad (32)$$

For the second term, we swap $\mu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$:

$$-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\phi(x) = \hbar^2\left[x^\rho\left(g^{\sigma\mu}\partial^\nu\phi + x^\mu\partial^{\sigma\nu}\phi - g^{\sigma\nu}\partial^\mu\phi - x^\nu\partial^{\sigma\mu}\phi\right)\right] \quad (33)$$
$$-\hbar^2\left[x^\sigma\left(g^{\rho\mu}\partial^\nu\phi + x^\mu\partial^{\rho\nu}\phi - g^{\rho\nu}\partial^\mu\phi - x^\nu\partial^{\rho\mu}\phi\right)\right] \quad (34)$$

Adding these two terms, we see that all the second derivative terms cancel, and since $g^{\mu\nu} = g^{\nu\mu}$, we can group terms to get

$$\left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(x) = i\hbar\frac{\hbar}{i}\left[g^{\nu\rho}\left(x^\sigma\partial^\mu - x^\mu\partial^\sigma\right) + g^{\nu\sigma}\left(x^\mu\partial^\rho - x^\rho\partial^\mu\right)\right] \quad (35)$$
$$+i\hbar\frac{\hbar}{i}\left[g^{\mu\rho}\left(x^\nu\partial^\sigma - x^\sigma\partial^\nu\right) + g^{\mu\sigma}\left(x^\rho\partial^\nu - x^\nu\partial^\rho\right)\right] \quad (36)$$

Comparing this with 18 we find

$$[\phi,[M^{\mu\nu},M^{\rho\sigma}]] = i\hbar\left(g^{\nu\rho}[\phi,M^{\sigma\mu}] + g^{\nu\sigma}[\phi,M^{\mu\rho}] + g^{\mu\rho}[\phi,M^{\nu\sigma}] + g^{\mu\sigma}[\phi,M^{\rho\nu}]\right) \quad (37)$$
$$= i\hbar\left[\phi,\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \left(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}\right)\right] \quad (38)$$

where we've used the antisymmetry of $M^{\sigma\mu}$ and $M^{\rho\nu}$ to get the last line.

We thus find that

$$[M^{\mu\nu},M^{\rho\sigma}] = i\hbar\left(\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \left(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}\right)\right) + A \quad (39)$$

where $[\phi, A] = 0$, which agrees with 9, up to the possible factor A.