

## ENTROPY - A FEW EXAMPLES

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Post date: 12 July 2021.

The entropy of a substance is given as

$$S = k \ln \Omega \quad (1)$$

where  $\Omega$  is the number of microstates accessible to the substance.

For a 3-d ideal gas, this is given by the Sackur-Tetrode formula:

$$S = Nk \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi mU}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right] \quad (2)$$

where  $V$  is the volume,  $U$  is the energy,  $N$  is the number of molecules,  $m$  is the mass of a single molecule and  $h$  is Planck's constant.

Although this formula looks a bit complicated, we can see that increasing any of  $V$ ,  $U$  or  $N$  increases the entropy. For an isothermal expansion, the gas expands quasistatically so that its temperature stays constant. This means that  $U = \frac{3}{2}NkT$  also stays constant, so that only the volume changes. Since the gas is doing work  $W$  by expanding, the energy for the work must be provided by an amount of heat  $Q$  input into the gas to maintain the temperature as constant. This heat is given by the formula

$$Q = NkT \ln \frac{V_f}{V_i} \quad (3)$$

where  $V_i$  and  $V_f$  are the initial and final volumes.

However, from 2, the change in entropy in a process where only the volume changes is

$$\Delta S = S_f - S_i = Nk \ln \frac{V_f}{V_i} \quad (4)$$

Combining these two equations gives

$$\boxed{\Delta S = \frac{Q}{T}} \quad (5)$$

This relation is valid for the case where the expanding gas does work, so that heat must be input to provide the energy for the work. In a free

expansion, the gas expands into a vacuum so does no work (well, technically, after some of the gas has entered the vacuum area, it's no longer a vacuum so that *some* work is done, but we'll assume the vacuum area is very large so we can neglect this). In this case, the internal energy  $U$  still doesn't change, since the gas neither absorbs any heat nor does any work, so  $\Delta U = Q + W = 0$ . However, the volume occupied by the gas *does* increase (and it's the only thing that changes) so 4 is still valid, although 5 is not.

Another property of 2 is that if the energy  $U$  drops low enough, the log term can decrease below  $-\frac{5}{2}$  making  $S$  negative. This isn't possible, so the Sackur-Tetrode equation must break down at low energies. For a monatomic ideal gas,  $U = \frac{3}{2}NkT$ , so this implies that things go wrong for low temperatures. For example, suppose we have a mole of helium and cool it (assuming it remains a gas). Then the critical temperature is found from

$$-\frac{5}{2} = \ln \left( \frac{V}{N} \left( \frac{2\pi mkT}{h^2} \right)^{3/2} \right) \quad (6)$$

$$T_{\text{crit}} = \frac{h^2}{2\pi mk} \left( \frac{N}{V} e^{-5/2} \right)^{2/3} \quad (7)$$

If we start at room temperature  $T = 300$  K and atmospheric pressure  $P = 1.01 \times 10^5$  Pa, and can hold the density  $N/V$  fixed, this will give an actual temperature at which the entropy becomes zero. The density is

$$\frac{N}{V} = \frac{P}{kT} = 2.44 \times 10^{25} \text{ m}^{-3} \quad (8)$$

The mass of a helium atom is  $4 \times 10^{-3} \text{ kg mol}^{-1} / 6.02 \times 10^{23}$ , so plugging in the other values gives

$$T_{\text{crit}} = 0.012 \text{ K} \quad (9)$$

In fact, helium liquefies at around 4 K, so it appears that 2 might actually be valid for the region where helium remains a gas.

As a final example, we can observe that the entropy of an ideal gas is  $Nk$  multiplied by a logarithm, and of an Einstein solid is also  $Nk$  multiplied by a logarithm (because  $\Omega \approx (qe/N)^N$  for high-temperature solids). For any macroscopic object,  $N$  is a large number and the logarithm is much smaller, so for a rough order-of-magnitude estimate of the entropy, we can neglect the log term and take  $S \sim Nk$ . A few such estimates are:

For a 1 kg book, we can take it to be 1 kg of carbon, with a molar mass of  $12 \times 10^{-3} \text{ kg mol}^{-1}$ , so the entropy of a book is around

$$S \sim \frac{6.02 \times 10^{23}}{12 \times 10^{-3}} (1) (1.38 \times 10^{-23}) = 692 \text{ J K}^{-1} \quad (10)$$

[Although, the entropy increase in my brain from trying to understand some physics texts must be considerably more than this.]

For a 400 kg moose, which we can approximate by 400 kg of water with molar mass of around  $18 \times 10^{-3} \text{ kg mol}^{-1}$ , we have

$$S \sim \frac{6.02 \times 10^{23}}{18 \times 10^{-3}} (400) (1.38 \times 10^{-23}) = 1.85 \times 10^5 \text{ J K}^{-1} \quad (11)$$

For the sun, we can take it to be  $2 \times 10^{30}$  of ionized hydrogen (protons) with molar mass of  $10^{-3} \text{ kg mol}^{-1}$ . The entropy is around

$$S \sim \frac{6.02 \times 10^{23}}{10^{-3}} (2 \times 10^{30}) (1.38 \times 10^{-23}) = 1.66 \times 10^{34} \text{ J K}^{-1} \quad (12)$$

#### PINGBACKS

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