INTERACTING EINSTEIN SOLIDS - A FEW EXAMPLES

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Post date: 11 July 2021.

Here are a few more examples of the probabilities of various macrostates in two interacting Einstein solids. As before, we have two solids, A and B, containing N_A and N_B oscillators and q_A and q_B quanta of energy, with $q_A + q_B = q =$ constant. For any particular partition of the quanta, that is, for particular values of q_A and q_B, the total number of microstates available to the compound system is

$$\Omega_{\text{total}} = \Omega_A \Omega_B = \binom{q_A + N_A - 1}{q_A} \binom{q_B + N_B - 1}{q_B} \tag{1}$$

The overall number of microstates is

$$\Omega_{\text{overall}} = \binom{q + N_A + N_B - 1}{q} \tag{2}$$

Example 1. Consider a simple system with $N_A = N_B = 3$ and $q = 6$. Using Maple to calculate the binomial coefficients (Maple has a 'binomial' function that does this automatically) and produce the plot, we have

$$\Omega_{\text{overall}} = \binom{6+3+3-1}{6} = 462 \tag{3}$$

$$\Omega_A = \binom{q_A + 2}{q_A} \tag{4}$$

$$\Omega_B = \binom{q - q_A + 2}{q - q_A} = \binom{8 - q_A}{6 - q_A} \tag{5}$$

$$Prob(q_A) = \frac{\Omega_A \Omega_B}{\Omega_{\text{overall}}} \tag{6}$$

Plugging in the numbers, we get Table 1.

A bar chart of the probabilities is in Fig. 1.

As before, the most likely state is when the energy is equally distributed between the two solids.

q_A	Ω_A	Ω_B	Ω_{total}	$Prob(q_A)$
0	1	28	28	0.061
1	3	21	63	0.136
2	6	15	90	0.195
3	10	10	100	0.216
4	15	6	90	0.195
5	21	3	63	0.136
6	28	1	28	0.061

TABLE 1. Probabilities for 6 energy quanta and $N_A = N_B = 3$

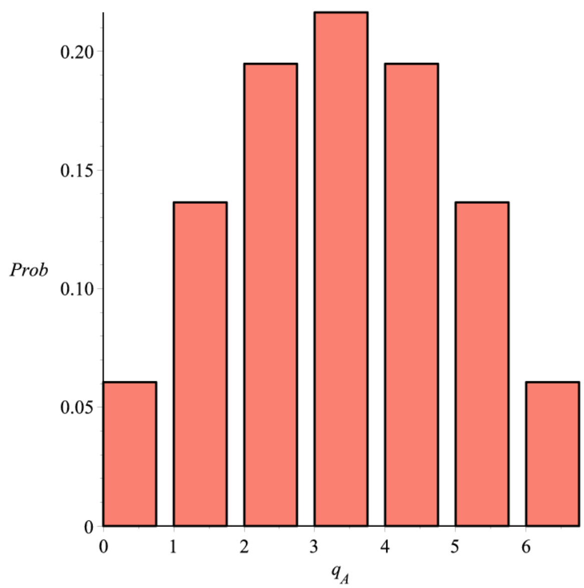

FIGURE 1. Probability distribution for 6 energy quanta and $N_A = N_B = 3$.

Example 2. Now we'll see what happens if one solid has more oscillators to store energy than the other one. We'll take $N_A = 6$, $N_B = 4$ and $q = 6$. We now have

q_A	Ω_A	Ω_B	Ω_{total}	$Prob(q_A)$
0	1	84	84	0.017
1	6	56	336	0.067
2	21	35	735	0.147
3	56	20	1120	0.224
4	126	10	1260	0.252
5	252	4	1008	0.201
6	462	1	462	0.092

TABLE 2. Probabilities for 6 energy quanta and $N_A = 6$ and $N_B = 4$.

$$\Omega_{\text{overall}} = \binom{6+6+4-1}{6} = 5005 \tag{7}$$

$$\Omega_A = \binom{q_A+5}{q_A} \tag{8}$$

$$\Omega_B = \binom{q-q_A+3}{q-q_A} = \binom{9-q_A}{6-q_A} \tag{9}$$

$$Prob(q_A) = \frac{\Omega_A \Omega_B}{\Omega_{\text{overall}}} \tag{10}$$

Plugging in the numbers, we get Table 2

A bar chart of the probabilities is in Fig. 2.

Since solid A has more oscillators, the probabilities are skewed towards more of the quanta being stored in solid A than in solid B.

Example 3. Now we'll ramp things up a bit and consider two solids with $N_A = 200$, $N_B = 100$ and $q = 100$. As there are 101 macrostates, we won't list all the various states. The formulas in this case are

$$\Omega_{\text{overall}} = \binom{200+100+100-1}{100} = 1.681 \times 10^{96} \tag{11}$$

$$\Omega_A = \binom{q_A+199}{q_A} \tag{12}$$

$$\Omega_B = \binom{q-q_A+99}{q-q_A} = \binom{199-q_A}{100-q_A} \tag{13}$$

$$Prob(q_A) = \frac{\Omega_A \Omega_B}{\Omega_{\text{overall}}} \tag{14}$$

The plot is in Fig. 3.

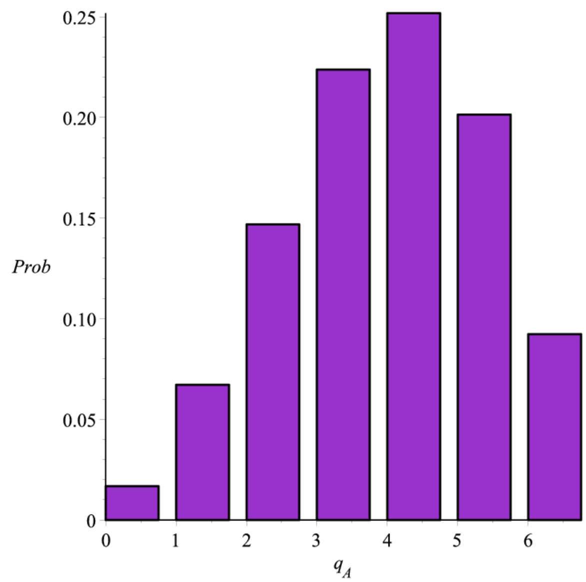

FIGURE 2. Probability distribution for 6 energy quanta and $N_A = 6$ and $N_B = 4$.

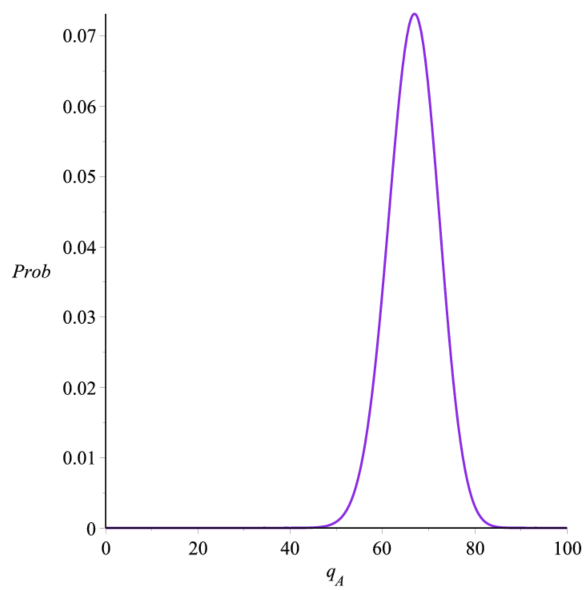

FIGURE 3. Probability distribution for 100 energy quanta and $N_A = 200$ and $N_B = 100$.

The maximum probability of 0.073 occurs at $q_A = 67$ and the minimum of 2.69×10^{-38} at $q_A = 0$. As solid A contains $\frac{2}{3}$ of the oscillators, the maximum probability is when $\frac{2}{3}$ of the energy is stored in solid A. Notice how vanishingly small is the chance of finding the system in a macrostate with anything less than about $q_A = 45$. Thus even though all microstates are equally probable, it is overwhelmingly likely that the energy will be more or less evenly distributed over all the oscillators.

PINGBACKS

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