INTERACTING EINSTEIN SOLIDS - RECTANGULAR PEAK IN MULTIPLICITY GRAPH

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In a system composed of two interacting Einstein solids, the multiplicity function, which gives the number of microstates as a function of the number of energy quanta q_A in solid A, is very sharply peaked about the point where the quanta are evenly distributed between the two solids. The width of the peak is approximately $q/\sqrt{N}$ for a system containing a total of q energy quanta and N oscillators.

We can get another, somewhat rougher, estimate of this width by first calculating the total number of microstates Ω_{total} accessible to the system (that is, the total number of microstates summed over all possible macrostates), and then finding the number of microstates Ω_{mp} in the most probable macrostate (even distribution of energy quanta). If we then assume that the peak in the graph is rectangular rather than Gaussian, then the width w of the peak is found from the area of the rectangular peak, according to

$$w = \frac{\Omega_{\text{total}}}{\Omega_{\text{mp}}} \tag{1}$$

The reasoning behind this is as follows. For large q and N, the system almost certainly has its energy quanta divided in a way that corresponds to a state within the peak on the graph. The area of this rectangular peak is the integral (technically a sum, but the points on the graph are so closely spaced as to make the graph essentially a continuous curve, so the sum of all states is effectively an integral) of the peak, which is the area of the rectangular peak. This integral is the total number of microstates, or Ω_{total}. The height of the peak is the number of microstates in the most probable macrostate, which is Ω_{mp}. Thus the area of the rectangle is

$$\Omega_{\text{total}} = \Omega_{\text{mp}} w \tag{2}$$

To apply this, let's consider a simple system where the two solids each have N oscillators and the total number of quanta is $q = 2N$. For large N and q, we can use the approximation derived earlier for the number of microstates in a solid with q quanta and n oscillators (I'm using a lowercase n here to distinguish it from the N in the problem):

$$\Omega \approx \sqrt{\frac{n}{2\pi q(q+n)}} \left(\frac{q+n}{q}\right)^q \left(\frac{q+n}{n}\right)^n \tag{3}$$

To find Ω_{total}, we can combine the two solids into one composite solid since we're interested in *all* microstates, no matter how the quanta are divided up between the two solids. In this case $q = 2N$ and $n = 2N$, so we get

$$\Omega_{\text{total}} \approx \sqrt{\frac{2N}{2\pi(2N)(4N)}} \left(\frac{4N}{2N}\right)^{2N} \left(\frac{4N}{2N}\right)^{2N} \tag{4}$$

$$= \frac{2^{4N}}{\sqrt{8\pi N}} \tag{5}$$

To find Ω_{mp}, we need to separate the solid into its constituent parts A and B, and assign quanta so that $q_A = q_B = \frac{q}{2} = N$. The total number of microstates for this particular macrostate is then

$$\Omega_{\text{mp}} = \left[\sqrt{\frac{N}{2\pi(N)(N)}} \left(\frac{2N}{N}\right)^N \left(\frac{2N}{N}\right)^N\right]^2 \tag{6}$$

$$= \frac{2^{4N}}{4\pi N} \tag{7}$$

The term in square brackets is Ω for one of the solids, since it contains $n = N$ oscillators and $q_A = N$ energy quanta. Since the other solid is identical, we just square the result to get Ω_{mp}.

The width of the peak is then

$$w = \frac{4\pi N}{\sqrt{8\pi N}} = \sqrt{2\pi N} \tag{8}$$

[The Gaussian width is $q/\sqrt{N} = 2\sqrt{N}$ and since $\sqrt{2\pi N} \approx 2.5\sqrt{N}$ the rectangular approximation isn't actually all that bad.]

The total number of macrostates in the two-solid system is $q+1 = 2N+1 \approx 2N$, so the fraction of macrostates that have reasonably large probabilities is

$$\frac{\sqrt{2\pi N}}{2N} = \sqrt{\frac{\pi}{2N}} \tag{9}$$

For a macroscopic solid with $N = 10^{23}$, this fraction comes out to around 4×10^{-12}. This shows just how unlikely it is that a macroscopic system will ever be found with an energy distribution significantly different from the most probable case.

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