PARAMAGNETS AND COIN FLIPS - PEAK AND WIDTH OF MULTIPLICITY FUNCTION

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For a couple of examples of Stirling's approximation we'll return to the two-state paramagnet. First, suppose we have a paramagnet with $N=10^{23}$ dipoles and that exactly half are in the spin-up state, so that $N_{\uparrow}=\frac{1}{2}\times10^{23}$. The number of microstates in this system is

$$\Omega = \frac{N!}{(N/2)!\,(N/2)!} \quad (1)$$

$$\approx \frac{\sqrt{N}N^{N}e^{-N}}{\sqrt{2\pi}\,(N/2)\,(N/2)^{N}\,e^{-N}} \quad (2)$$

$$= \frac{2^{N+1}}{\sqrt{2\pi N}} \quad (3)$$

$$\approx 2^{N} \quad (4)$$

where the last line is true for N values around 10^{23} since this makes 2^{N} a *very* large number, so that dividing by the square root won't change things much. [Note that in this approximation, Ω for the peak is actually equal to the total number of microstates summed over *all* macrostates, so the result says that an equal distribution is virtually certain.]

If the system was in a different microstate a billion times per second over a period of ten billion years, it would explore

$$10^{9}\times10^{10}\times365.25\times24\times3600=3.16\times10^{26} \quad (5)$$

microstates. The number of available microstates is

$$2^{10^{23}}\approx10^{3\times10^{22}} \quad (6)$$

so the actual number of microstates visited in 10 billion years is a vanishingly small proportion of those available. In theory, if we wait long enough, I suppose it's *possible* to visit all the "accessible" microstates, but we'd have to wait an almost infinite time (at least on a time scale of the current age of the universe) for this to happen.

If we flip N coins, the multiplicity function peaks sharply at a number N_h of heads equal to $N/2$. The actual multiplicity function in this case has the value given in 1, so this is the height of the peak.

To get an estimate of the width of the peak, we can define

$$x \equiv N_h - \frac{N}{2} \tag{7}$$

to be the deviation of the observed number of heads from the most probable value of $\frac{N}{2}$. The multiplicity function in terms of x is therefore

$$\Omega(x) = \frac{N!}{\left(\frac{N}{2}-x\right)!\left(\frac{N}{2}+x\right)!} \tag{8}$$

$$\approx \frac{\sqrt{N}N^N}{\sqrt{2\pi}\sqrt{\frac{N}{2}-x}\sqrt{\frac{N}{2}+x}\left(\frac{N}{2}-x\right)^{\frac{N}{2}-x}\left(\frac{N}{2}+x\right)^{\frac{N}{2}+x}} \tag{9}$$

$$= \frac{\sqrt{N}N^N}{\sqrt{2\pi}\sqrt{\left(\frac{N}{2}\right)^2-x^2}\left(\left(\frac{N}{2}\right)^2-x^2\right)^{\frac{N}{2}}\left(\frac{N}{2}-x\right)^{-x}\left(\frac{N}{2}+x\right)^{x}} \tag{10}$$

We can now work with logarithms and consider the case of large N, so we can neglect the square root terms. Thus the approximation becomes

$$\Omega(x) \approx \frac{N^N}{\left(\left(\frac{N}{2}\right)^2-x^2\right)^{\frac{N}{2}}\left(\frac{N}{2}-x\right)^{-x}\left(\frac{N}{2}+x\right)^{x}} \tag{11}$$

$$\ln\Omega = N\ln N - \frac{N}{2}\ln\left(\left(\frac{N}{2}\right)^2 - x^2\right) - x\ln\frac{\frac{N}{2}+x}{\frac{N}{2}-x} \tag{12}$$

$$= N\ln N - \frac{N}{2}\ln\left[\frac{N^2}{4}\left(1-\frac{4x^2}{N^2}\right)\right] - x\ln\frac{1+\frac{2x}{N}}{1-\frac{2x}{N}} \tag{13}$$

$$= N\ln N - \frac{N}{2}\ln\left(\frac{N}{2}\right)^2 - \frac{N}{2}\ln\left(1-\frac{4x^2}{N^2}\right) - x\left[\ln\left(1+\frac{2x}{N}\right) - \ln\left(1-\frac{2x}{N}\right)\right] \tag{14}$$

$$= \ln 2^N - \frac{N}{2}\ln\left(1-\frac{4x^2}{N^2}\right) - x\left[\ln\left(1+\frac{2x}{N}\right) - \ln\left(1-\frac{2x}{N}\right)\right] \tag{15}$$

We can now use the approximation $\ln(1+z) \approx z$ for $|z| \ll 1$ and get

$$\ln\Omega \approx \ln 2^N + \frac{N}{2}\frac{4x^2}{N^2} - x\left(\frac{2x}{N} - \left(-\frac{2x}{N}\right)\right) \tag{16}$$

$$= \ln 2^N - \frac{2x^2}{N} \tag{17}$$

Thus the multiplicity function near the peak is approximately

$$\Omega(x) \approx 2^N e^{-2x^2/N} \tag{18}$$

Note that for $x = 0$, we get the peak height in 4. [If we'd retained the square root factors in 10, we'd get the $\sqrt{2\pi N}$ factor back.]

Thus the peak falls to $1/e$ of its peak value when

$$x = \sqrt{\frac{N}{2}} \tag{19}$$

so the width of the peak is twice this, or

$$w = \sqrt{2N} \tag{20}$$

If we flipped 10^6 coins, a result of 501,000 heads is just outside the peak, since $x = \sqrt{\frac{10^6}{2}} \approx 700$, so this wouldn't be a particularly surprising result. However, a result of 510,000 heads is well outside the peak and *would* be a surprising result.

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